ELECTRICAL ENGINEERING

Basic Analysis

BY

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PREFACE

This text like many others has evolved from the introduction in a college curriculum of a new course for which existing texts were not suitable. In mimeographed form the text has been through several editions with revisions and expansions guided by the reaction of students and teachers and by much thought and discussion among the staff concerned.

Although the text represents no radical departure from the conventional, it has been the purpose of the author to lead the beginning student of electrical engineering into the subject with as much zest and interest as feasible, yet also with more than the usual attention to accuracy of statement and unambiguity of meaning.

It is important that *elementary* be not confused with *superficial;* that the initial concepts be sound and as free from the necessity of later revision as is possible with present knowledge of facts and without unreasonable complication. It is especially important that statements and concepts be avoided which allow and invite the student to infer untruths. With this in mind the author has attempted, nevertheless, to avoid regimenting the study into a "grind" of uninspiring "finger exercises"; the interest factor, especially in an introductory study, is of major importance. The kind and sequence of subjects utilized as vehicles for the basic facts, as well as the more informal style of presentation, are chosen with this objective.

Above all, the text purposes to encourage analytical thinking and to discourage the mere *memorizing* of descriptive material which in his precollege experience the student is too often led to believe is synonymous with "learning" or "studying." The questions and problems, of course, comprise a major element in this endeavor and the text cannot fulfil its purpose without liberal usage of them. Although not as numerous as might be desired, they will be found to be very largely original and highly purposeful as media for developing analytical study habits. It has been found preferable that the student actually *write* the question answers, partly to assist review, and partly because he so generally needs practice in crystallizing his rather vague thoughts into concise technical exposition.

The experienced teacher will find in this work numerous departures from the points of view usual to texts in this field. The author presents these in the belief that they convey more clearly to today's student old

truths too long presented in the words of yesterday's pioneers. The purpose of the unusual material of articles 16·10 to 16·12 may not be apparent to one who has not had previous experience with it. It has been found that the preparation of the student commonly is inadequate to permit him to approach this material from the relatively simple Kirchhoff law method given here; that he just is not in the habit of thinking about these problems in the same terms with non-capacitive circuits. The student then may be expected to have trouble here in proportion to his need for clearing up what the author considers a major deficiency in his understanding of capacitors and their behavior in electric circuits.

Attention should be called to the detachable work sheets in the back of the text. Certain graphs among the problems are reproduced here to conserve for the student the time for copying them from the text. Here also is provided a detachable copy of the *B-H* curves most used in problem solutions. They are intended for collection and re-issue by the instructor in examinations where the text itself is not permitted.

It has often been expressed among teachers in this field that the separate study of direct and alternating currents successively is a chronological consequence of development rather than the best teaching sequence. Some have gone so far as to assert that direct current should be taught after and as a special case of alternating current. Without subscribing to this extreme, the author does feel that it is unwise wholly to defer study of alternating current until a first course in electrical engineering is completed. This belief is based primarily upon consideration of the factors of relative *interest* and *difficulty* together with the desirability of allowing as much time as possible for the student to become acquainted with the alternating-current aspects of his fundamentals before extensive application in specialized studies. To mix the two, however, does require some care. Although the arrangement here provided is but one of many possibilities, it has been found to attain the objectives of improved interest and balance without destroying proper gradation of difficulty.

Mathematics through calculus and one year of physics are prerequisite to the study of this material. Knowledge of both is exercised throughout the work and is essential to the sequence and type of treatment here employed. Where the usual mathematics course for engineers is not given until the second year, the first term of mathematics should suffice to permit successful use of the text in the second term concurrently with the second term of calculus.

It is manifestly impossible to acknowledge separately the innumerable sources, many unknown, of these well-established basic facts in the

field of electrical engineering. For the numerous original items of concept and presentation which have survived the test of teacher and student, I am greatly indebted to the inspiration and enthusiastic cooperation given me by the several classes of students who have studied with me throughout the mimeographed status of this material, and to most of the staff of the school. In particular I am indebted to those of the staff who have participated with me both in teaching the material of the text and in the many hours of discussion which have insured its teachability and its soundness.

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CHAPTER I

INTRODUCTION

1·1. Electrical Knowledge. There is in the popular mind and in the minds of many technically educated men in fields other than electrical a feeling that electricity is much more intangible than the steam and water, the wind, the steel and stone, and a thousand other commodities with which the engineer plies his profession. Because they cannot directly see, or hear, or feel, or smell, or taste electricity it asks a bit more than their faith and imagination can cope with. They say "You don't really know what the stuff is! How can you really understand what you are doing with it?"

Have you paused to consider how little we really know about what anything actually *is*? Now do not confuse *knowledge* with *familiarity.* We are prone to develop a sense of knowledge because of everyday association, because we can identify and even classify this and that. But if a persistent inquirer asks, "What is it?" and says, "Stick to the question," we soon must admit, "I don't know."

Whether electrical or not, an engineer is primarily concerned not so much with what things *are* as with what they *do*. Of course he is interested in the identity of materials and in the developing of new materials. His work, however, is in the application of these materials with the purpose that they *do* a particular job. It is our purpose here to embark upon an orderly examination of the most useful elements of the knowledge that has so far been disclosed regarding the electrical characteristics of materials in various modes of combination and association. This knowledge is best known as "fundamental" or "basic" knowledge —sometimes as "theory," although not at all in the sense of theoretical or unpractical. One of the distinctive characteristics of a good engineer is his ability to strike an *economic balance* between purely *mental idealism* and purely *physical accomplishment.* He must conceive, and plan, and formulate but he must *achieve* the physical objective of that plan within *economic limits.* His feet must be on the ground. He cannot be impractical.

1·2. The Analog. It is sometimes helpful to portray the characteristics of an unknown by likening it in part to one or more knowns. We do this in describing one person to another: "In this respect he is like

John but in that he behaves like Peter, etc." The difficulty with description by analog is to determine just how far it holds *exactly*, to know where it breaks down. The analog is often useful in capturing a first glimmer of understanding but may then better be discarded as a swimmer would discard the water wings of his first attempts. Like the swimmer too he may do better without water wings at any time. The analogs here presented are best "taken or left" as the individual reader finds beneficial.

1·3. Fundamentals. Engineering is a scientific profession and *electrical* engineering is no exception. It is not practiced successfully by rule of thumb, by intuition, or by applying generous "good-luck" factors. Exact knowledge is essential. Our knowledge of what electricity will do has been obtained largely by constructing a great variety of sets of planned circumstances or "situations" in which it may perform—situations carefully controlled and repeated until no reasonable doubt remains as to the truth or significance of the evidence.

Fortunately these researches show that, as the circumstances become more *complicated*, the behavior is generally consistent with that observed in the *simpler* cases. For this reason it becomes possible to work out what might be called "behavior patterns" which constitute an assembly of the simple behavior elements. The simple behaviorisms then take on a special importance and are looked upon as the basic ingredients or fundamentals of electrical technology.

Electrical engineering involves the reliable prediction of what will happen under a complication of circumstances not previously observed in duplicate. Such prediction is possible only because the situation can be analyzed or split up into a combination of the simpler fundamental components about which reliable knowledge has been established.

The engineering method of approach to a problem then comprises:

1. Determining the really fundamental causative elements involved in its make-up and the pattern of their relationship.
2. Integrating or summing up these causative elements to determine the individual and mutual results which they are known to produce.
3. Devising some kind of check on the predictions. This may take the form of a different concept of the elements and pattern composition in (1) or may require experiment with some kind of model.

1·4. Mathematics in Engineering. It is fortunate that nature is for the most part not only consistent but also orderly. Let us consider the case of heat power produced by electric current in a resistor. The evidence obtained from a sufficiently extensive observation of currents traversing an unvarying resistor discloses a consistent orderliness about the phenomenon which we call "natural law." More current gives more

heat power. We could arrange the experimental data in tables so that, for a given resistor, the exact value of heat power for a particular value of current could be looked up. We might plot the data, constructing a graphical chart from which the particular value of heat power could be taken.

What we actually do is to observe that the law for this particular phenomenon is most concisely expressed in the language and symbolism of mathematics: ***The heat power is proportional to the square of the current*** or $P = RI^2$. It is not uncommon that, as in this case, we find the experimentally established law in the repertoire of the standard *functions* of mathematics. If not, and if the law finds sufficient application, we sometimes *create* for it a function new to the lore of conventional mathematics.

Mathematics not only serves the engineer in this way to simplify his "bookkeeping" but also offers its processes to expedite his analysis of a complex situation into its fundamentals, or his assembly of the fundamentals to synthesize the situation. Mathematics thus constitutes a *tool* with which the engineer can *apply* his laws of nature or fundamentals to the study and solution of his technical problems.

It is imperative that the student obtain a clear concept of what mathematics can and cannot do for him. For him it is a versatile instrument or machine which he may set up to take certain "raw material" and process it according to certain specifications. It is futile for him to put his raw material into the machine until he has the "specifications" or fundamentals in full, and has the machine set up to process in accord with the specifications; otherwise the product of his machine is meaningless. In other words, *he wastes his time to employ mathematics merely for juggling basic relations on the long chance that somehow, eventually, the desired product must pop out of it.* Mathematics is not more clairvoyant than its operator!

It often happens that the specifications are not so rigorous as to enable the mathematical "machine" to produce *the* answer. Several "answers" may come out. Means for recognizing the pertinent answer vary with the particular case.

It is quite likely that no available mathematical processing will exactly fit the natural laws of the actual phenomenon. The mathematically correct answer may then be too far from the true answer to be useful. For pedagogical reasons teachers and texts of engineering are inclined to overidealize the brutal facts of actual phenomena, that is, to ignore or dismiss discrepancies between the physical and mathematical which may produce really serious error. The student should strive to develop a wariness of such pitfalls—to remember that the

machine may be ignorant of the apparent dictates of misfit or over-idealistic mathematics.

It is highly important that the student come to evaluate the place of mathematics in his profession—its practical indispensability, yet its utter subordination to physical reality.

1·5. Formulas—Memorizing Them. There is a popular and widespread misconception, even in pre-college schools, that the study of engineering involves primarily the memorizing of the "magic formulas" which engineers use and the practice of their application until the student is proficient enough to hold a job. Many of us, before our teens, have looked worshipfully at the locomotive engineer and have visualized all engineers in terms of this one who then came within our realm of experience and comprehension. ***The formula concept*** of the engineer is no less erroneous than our teen concept and ***must be outgrown*** before successful study of this work is possible.

What is a formula? It is largely a *recipe*, a manufacturing instruction in concentrated form. It may come from the application of fundamental knowledge or it may be the result of cut-and-try experience. In general formulas are "formulated" by mathematical process from two or more relatively simple fundamental relations or natural laws. ***Empirical formulas*** are the direct result of quantitative experiment without the direct use of basic laws or relations. They are commonly used when the phenomenon defies analysis in terms of basic quantitative relations; they are created by matching portions of the curves of the known functions of analytical geometry to the graphically plotted experimental data.* Rarely does an engineering formula come from pure accident or good luck.

The professionally trained engineer is expected to qualify as a *creator* more than as a user of formulas. He is not merely a formula technician or recipe operator. He is concerned with the analysis of a problem into its basic components and the solution of the problem by means of his understanding of basic concepts and their interrelations as symbolized in fundamental relations or natural laws.

Observe that we are not indicating that the engineer need not know the fundamental relations (as distinguished from formulas) so that he can write down the symbolism for them as readily as his own initials. The point is that the memory process must not be one of *rote* memory, as required for an isolated name or a registration number. It has been amply and grievously demonstrated that to pursue the rote process is a fatal mistake even though it may in specific instances seem effective.

* See Lipka, "Graphical and Mechanical Computation," New York, John Wiley and Sons, 1918.

It cannot be too strongly emphasized that ***fundamental relations must be learned by gaining understanding of the phenomenon*** for which the relation is mere quantitative symbolism, and by understanding it so well that the relation becomes obvious from the concept of the phenomenon without invoking the aid of rote memory. Ohm's law, for example, is too often said to *be* $E = RI$. At best this relation is but one form of a symbolism for the law. Simply to memorize this symbolism is a task of few minutes, but to state the law in good, concise, foolproof technical English and to make numerous applications of the law is quite another task. It is this latter task, however, which is typical of the study in this text.

1·6. Basic Concepts. To establish the exact elements which constitute the foundation of our physical world is by no means as simple as might at first be presumed. We are quite certain of the basic elements as a group, but which are most basic as compared with others is controversial. ***Matter, energy, time,*** and ***space*** are for most of us so basic that our understanding of how they might be analyzed into more fundamental entities is perfectly blank and we simply must accept them as truly basic. Nevertheless we are obliged to admit that the four are not independent entities. Kinetic energy, for example, is related to the mass of matter and to velocity which is time rate of traversing space: $W = \frac{1}{2}mv^2$. If we were really to get at the bottom of it all, we could hardly accept dependency among our basic entities any more than we could permit it in a set of simultaneous algebraic equations which provide the basic information for solution of a given problem in algebra. Asked which of the four we would drop, we would be hard put to it for a rational answer. Nevertheless, because a solution may seem not to be just around the corner is no excuse for not making efforts toward solution. We at least may orient our concepts and organize the relations among them more fully than is possible by piecemeal efforts to study just "these" and "those" as a particular occasion may of itself require.

The author has conceived and developed a graphical structure which provides an illustration of the kind of organizing effort just advocated for the reader. It is important to observe that this is but an illustration and makes no claim to represent the ultimate of such endeavors. Figure 1·1 is for mechanical quantities. The basic entities *energy* (W), *momentum* (M), *displacement* (L), and *time* (T) are represented with energy at the center on the premise that energy may well be the entity of supreme import. The symbolism and the relations follow a very simple pattern: Any pair of quantities attached to opposite sides of a *circled* quantity by equal lengths of a continuous line (straight or curved) are

factors whose *product* equals the circled quantity. These are tabulated as follows:

W	P	L	M
PT FL MV $\frac{1}{2}mV^2$ $\frac{1}{2}SF^2$	FV	VT FS	FT mV

W = energy
P = power
L = displacement distance
M = momentum
V = velocity
F = force
T = time
m = mass
S = stiffness

Given three quantities W, L, and T, to be considered the basic ingredients, all other quantities on the diagram are derived from these. Very often they are derivatives in the calculus sense. For example:

$$V = \frac{L}{T}, \text{ or in calculus } \frac{dL}{dT}$$

then

$$M = \frac{W}{V}, \text{ or in calculus } \frac{dW}{dV}$$

also

$$F = \frac{W}{L}, \text{ or in calculus } \frac{dW}{dL}$$

The notion that force F might be a derived quantity dW/dL is likely to arouse some skepticism because we commonly feel that it is one quantity which is so real and tangible that with little thought we presume that it must be as basic as it is tangible. We are more familiar with the point of view that $W = \int FdL$. The possibility that force may be merely one of the tangible manifestations of energy, however, seems worthy of consideration in an analysis that leans more to the philosophical than to the dogmatic. It is permissible to write $F = dW/dL$ and we may define F as the rate of change of energy in a body with respect to its displacement, i.e., ***space rate of energy displacement.***

It is interesting to observe that the graph suggests that *momentum M* is more basic than mass m. This is further supported by the relativity theory which recognizes that mass is a function of velocity and therefore not one of the independent basic entities. It would seem desirable to emphasize more fully than is the practice in elementary studies of kinetics the basic importance of momentum in comparison with mass; that in the "dance of matter" mass is but a "wallflower."

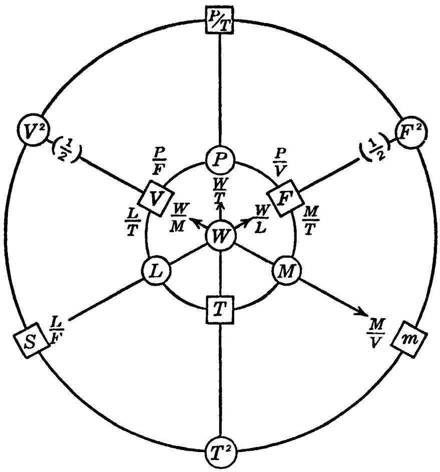

FIG. 1·1. A graphic coordination of relations among basic mechanical quantities.

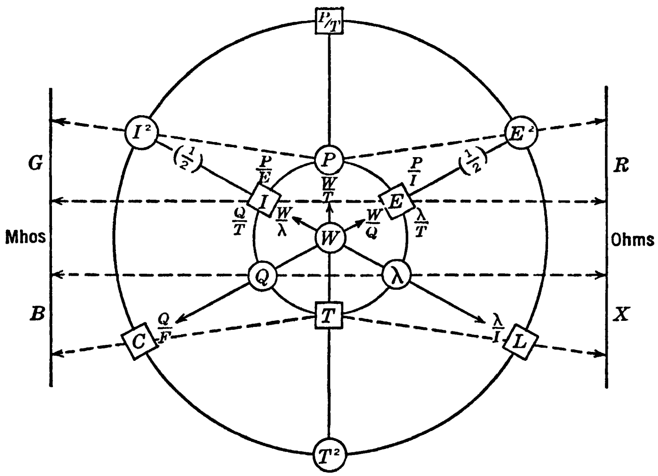

FIG. 1·2. A graphic coordination of relations among basic electrical quantities.

The corresponding graphical organization of electrical entities is shown in Fig. 1·2 and is tabulated below. Although this will mean more to

W	P	Q	λ	Ohms	Mhos
PT	EI	IT	ET	$\frac{P}{I^2}$	$\frac{P}{E^2}$
QE		CE	LI	$\frac{E}{I}$	$\frac{I}{E}$
$I\lambda$				$\frac{L}{T}$	$\frac{C}{T}$
$\frac{1}{2}LI^2$				$\frac{\lambda}{Q}$	$\frac{Q}{\lambda}$
$\frac{1}{2}CE^2$					

W = energy
P = power
Q = electron or charge displacement
λ = magnetic flux linkage
I = current
E = emf or voltage
T = time
L = inductance
C = capacitance
R = resistance
X = reactance
G = conductance
B = susceptance

the reader after he has proceeded considerably further with his study of electricity in these chapters, it may serve here to suggest some of the analogous quantities in mechanical and electrical engineering and the greater completeness of concepts in the latter.

It will be noted that the vertical axis is the same for the two figures. This of course results jointly from the universality of time (T) and the convertibility of energy (W) from mechanical to electrical and vice versa. The correspondence between Q of Fig. 1·2 and L of Fig. 1·1 may not be evident until it is appreciated that each represents an interpretation of *displacement* in its own realm.

While the weak position of resistance and its reciprocal, conductance, beyond the "side lines" of the diagram may at first seem disconcerting it is consistent with the weakness of the premise that they are simple constants. This will be discussed in later chapters.

It is particularly important that there be here no misunderstanding about the quantities velocity V and current I which appear in analogous positions on the two diagrams. Unfortunately for our purpose the term *current* is used popularly to mean *velocity*. When we say of a river, "The current is swift," we mean only that the velocity of the water is high. Technically speaking, current measures the time rate of displacement of substance. For water this is commonly in gallons per minute. Clearly the gallons per minute of flow, or current, for a large sluggish river may greatly exceed that for a swiftly flowing brook. For a given current of inelastic material, the average velocity is inversely proportional to the section normal to the displacement. ***Electric current is***

time rate of displacement or "flow" of electricity (usually measured in coulombs per second) and it will later be developed that the *velocity* of displacement even for our largest electric currents is actually that of a sluggish river. High current does *not* mean high electron or charge velocity.*

Another term which is also confused frequently because of a difference in technical as compared to popular meaning is "power." Power is popularly used vaguely to imply either force or potentiality for achievement. Technically, ***power is entirely restricted to mean time rate of displacement (or conversion) of energy,*** i.e., time rate of doing work. High power means *not* high *capacity* for doing work but high *time rate* of doing work. Graphically, ***energy is represented by the area under a power-time plot.*** On a plot of energy versus time, power is the slope of the curve at any instant. Energy is a *commodity* which can be bought and sold; power is simply the *speed of the transaction.*

1·7. Electrical Energy. Energy may be classified into at least four kinds according to the composition of matter:

Matter	Energy
Substance (body)	Mechanical
Molecule	Thermal
Atom	Chemical
Electron	Electrical

Unlike other forms of energy, ***electrical energy is a negligible natural resource.*** It not only has to be obtained largely by conversion from the other three, but also is of little use as such and must be reconverted before using. Why should this double conversion be employed more and more instead of a direct application of the energy as found in nature? The answer may be made evident by comparing the qualities of electrical energy with those of good currency; both are highly ***convertible,***

* The use of the term *current,* even electrically, customarily takes liberties with strict usage which cannot be ignored here. It is quite usual to find such expressions as "the current flows," "the flow of current," and the less objectional "current flow" when it would seem preferable to say "the current is directed," "the direction of current," or merely "the current." This seems to arise in part from a long period of indefiniteness about the substance of the current when the layman, at least, had little concept of electric current beyond its being "juice." He naturally assigned to "current" a sense of substance beyond the mere *rate of flow* which it strictly connotes. Because the direction of electron flow has been found to be opposite to that assumed for conventional current, it is still embarrassing to explain just *what* actually flows in the direction of the conventional current. It seems unwise to avoid here the accepted everyday language of the profession so long as it is clear to the reader who, sooner or later, can hardly avoid using it.

conveyable, and ***controllable.*** Electricity, like currency, is a *superior medium of (energy) exchange* and is useful largely for this reason.

Among the many applications for which electricity is superior because of one or more of these qualities, *communication* should have special mention; here in particular all three qualities are paramount and electricity reigns supreme.

1·8. Energy Computations. It follows that facility in computing the electrical equivalent of energy to and from other forms is important. The necessary conversion constants are available in nearly any handbook. Familiarity with the customary units of measure of each kind of energy is a prerequisite and care must be used to avoid leaving final results in such incongruous forms as "btu taken by an electric heater" or "watts output of a motor."

In general, energies are computed from what we have called the *ingredients* or *derivatives* of energy. In the case of mechanical energy these may be force and displacement. Problems involving the conversion of energy commonly involve a sufficient number of ingredients so that it is helpful to utilize *dimensional analysis.* This may be either to determine certain unknown factors or to check the correctness of relations obtained otherwise.

1·9. Conversion of Units. Care is required to avoid confusion in the conversion of quantities from one unit to another. The difficulty arises from failure to distinguish clearly between a mere *statement* of the relative size of units and an *algebraic equation* involving equality between a quantity expressed in one system of units and the same quantity expressed in another system of units. For example, the statement, 12 in. equals 1 ft, is not an algebraic equation although it frequently is written 12 in. = 1 ft. In algebraic form the same relation is

$$x = 12y$$

where x is in inches and y is in feet. The conversion factor 12 clearly appears on the *inch* or on the *foot* side of the equality sign according to which of the above two expressions is used.

The author recommends that the reader always endeavor to fix in mind which of the two units in question is the *smaller.* It then must require *more* of these units to represent a given quantity. Because the inch is smaller than the foot, a given length must be expressed by a larger number of inches than of feet. Once this is clear, the procedure required to accomplish it is obvious. Simply to remember that we "multiply feet by 12 to get inches," may seem attractive but it is safer and more in line with engineering thinking to ***visualize the size of the unit*** and act accordingly.

A list of selected constants is provided by Table I in Appendix I. More complete listings are available in any of several handbooks.

1·10. Systems of Units. There are three qualifications which a system of units should meet to win the preference of potential users.

1. The *size* of the units should permit reasonable size numbers in the field for which they are intended, e.g., inches or centimeters for shaft diameters, but miles or kilometers for terrestrial distances.
2. The *relative size* of the units should permit their use in the maximum number of important fundamental relations without need for compensating conversion constants or "nuisance factors," e.g., power = emf × current where watts = volts × amperes; not power = speed × torque where HP = k (rpm) × (lb-ft). The conversion factor $k = 2\pi/33{,}000$ is required only by reason of the blind choice of the HP unit for power.
3. The *multiples* and *subdivisions* of each unit should be on a *decimal* basis such as that of the metric units, e.g., 1 cm = 0.01 meter, 1 km = 1000 meters; not 1 ft = 12 in., 1 yd = 3 ft, 1 ton = 2000 lb.

The English systems of units, for the most part, qualify only for the first of these requirements. The metric systems of units fulfil requirements (1) and (3), and to a considerable extent also (2). It is for this reason that the CGS system, familiar to all students of physics, is popular among scientific workers. In engineering work the practical difficulty of changing the equipment and personnel of industry from the well-established English system requires that the engineer be familiar with both systems in order to deal with both scientific and industrial workers and their literature as required by the nature of his work.

In the same way that a student of languages usually must begin with a native tongue and continue to think in terms of this language by the process of translating, so a technical student acquires facility in thinking in terms of a particular system of units which becomes his "native tongue" and into which, for some time, he must convert other units before he can feel confident about his work with them.

While there is no system of units which entirely fulfils the foregoing three requirements for general engineering use, the International Electrotechnical Commission (IEC) at a meeting in 1935 adopted a modified CGS system known as the MKS (meter-kilogram-second) system which has been urged and accepted by many leaders in the engineering profession as the best yet to achieve international sanction. For many purposes the basic units of the MKS system are of better size than the CGS units to meet requirement (1). This is especially true for the kilogram unit of mass, the newton unit of force, and the joule unit of

energy as compared with the corresponding CGS gram, dyne, and erg units which are impractically small for power engineering quantities.

From the electrical point of view the MKS system has several advantages over the CGS system which will be developed by its use in this text. These are typified by the use of the volt, ampere, ohm, watt, and joule units of industrial practice instead of the annoying abampere, erg, etc., units required by the CGS system to avoid conversion factors in basic relations. Except where expressly otherwise indicated, ***all relations developed in this text apply without conversion factors when MKS units are used.***

1·11. Efficiency. In popular language the term *efficiency* is used mostly in a rather vague qualitative sense to indicate ***efficacy.*** In technical parlance, efficiency has a very definite and restricted quantitative meaning. ***The efficiency of a device is the ratio of the useful output from it to the total input into it.***

Unless otherwise indicated it is understood to mean the ratio of *power* output to *power* input for a *specified* condition of operation. With few exceptions the input and output will be expressed in the *same* unit of measure so that the ratio may be expressed in *per cent* without reference to the particular unit used. One such exception is found in electric lamps where it is more convenient to leave the output in lumens and the input in watts, the efficiency being then in lumens per watt. Even then the efficiency is, of course, dimensionless (the lumen is a power unit).

It often happens that equipment is not worked continuously at full rated output. Distribution transformers, for example, commonly are obliged to put out just what the customers demand and the demand fluctuates widely throughout the day and hour. The efficiency of the transformer fluctuates with the load, and instantaneous or power efficiency does not suffice to indicate the performance over a period of time. For this purpose we use energies (integrated power) for the efficiency ratio and call it ***energy efficiency.*** The time period is usually 24 hours and gives rise to the term ***all-day efficiency.***

The energy output and the energy input, when electrical, are commonly available from watthour (or kilowatthour) meter measurements. When the metered values of output and input energies are not available it is necessary to compute the energy efficiency from a power-time graph (or equivalent data) together with the power efficiencies concerned. Given a graph of the power output and of the power input versus time for the desired period, the energy efficiency is simply the ratio of the area under the output curve to the area under the input curve. The following example is typical of the nongraphical problem.

Example. A 100-kw transformer operates on the following schedule:

Full load output for 8 hrs. at 96% efficiency
$\frac{1}{2}$ full load output for 6 hrs. at 75% efficiency
$\frac{1}{10}$ full load output for 7 hrs. at 35% efficiency
No load output for 3 hrs. with 3 kw input

Compute the all-day efficiency.

Solution. Total energy output is

$$W_o = 100 \times 8 + \tfrac{100}{2} \times 6 + \tfrac{100}{10} \times 7 = 1170 \text{ kwh}$$

Total energy input is

$$W_i = \frac{100 \times 8}{0.96} + \frac{100 \times 6}{2 \times 0.75} + \frac{100 \times 7}{10 \times 0.35} + 3 \times 3 = 1442 \text{ kwh}$$

All-day efficiency is

$$\eta = \frac{W_o}{W_i} = \frac{1170}{1442} = 81.0\%$$

It is to be noted expressly that energy efficiency is *not* an arithmetic average of the values of *power* efficiencies each weighted according to its duration. In fact a comparison of the mathematics involved readily indicates that energy efficiency is not obtainable by any kind of a mean of the power efficiencies.

Energy efficiencies, like power efficiencies, are not always expressed in per cent. The overall energy efficiency of a steam-electric power plant, for example, is seldom expressed in per cent but rather in kilowatthours output per pound of coal.

When the efficiency is known for each of a series of devices which in chain fashion convey energy from a given source to an ultimate receiver the ***overall efficiency*** of the whole series of devices is simply the product of the individual component efficiencies. This is equally true for power and energy efficiencies and of course is due to the simple fact that the output of each component device is the input for the next successive device.

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2. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., Seventh Edition, Section 1.
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QUESTIONS

1·1. (*a*) Distinguish explicitly between power and energy.

(*b*) Given a plot of power versus time, explain how to find energy in foot-pounds when power is in horsepower and time is in minutes.

1·2. Distinguish clearly between the concepts of *velocity* and *flow.*

1·3. Name three forms of energy, give a unit of measure of each and the relation (or conversion constant) between each pair.

1·4. Electrical energy is not an appreciable natural resource like water power or fuels. Account for the growth in use of energy in this form.

1·5. What properties of energy in electrical (not electromagnetic) form account for its supremacy in communication?

1·6. What source of confusion is encountered in converting a quantity from measure in one system of units to measure in another system of units? What method is suggested for avoiding such confusion?

1·7. Explain clearly the distinction between *power* efficiency and *energy* efficiency.

1·8. Give the quantities, e.g., mass, time, power, which the following units measure: (*a*) joule, (*b*) kilowatthour, (*c*) calorie, (*d*) gram, (*e*), gram-calorie, (*f*) watt, (*g*) lumen, (*h*) ohm, (*i*) farad, (*j*) radian.

PROBLEMS

1·1. An express elevator and load, weighing 3000 lb, traverses the entire 450-ft height to the roof of a building in 3 minutes. The motor which runs the elevator is 85 per cent efficient during the run and 30 per cent of its output is lost in mechanical friction of gearing, bearings, windage (air friction), etc. The motor speed is 1760 rpm.

(*a*) What is the overall efficiency?

(*b*) What is the average horsepower output of the motor?

(*c*) What average torque does the motor deliver?

1·2. A ferris wheel has 12 seats suspended from equidistant supports on a circumference of 20 ft radius. The maximum allowable load in each chair is 800 lb. The wheel makes one revolution per minute and requires a motor output of 5 hp at 1500 rpm to supply the power losses for any allowable loading.

(*a*) Compute the maximum torque required of the 1500-rpm motor when the wheel has the worst allowable unbalance of load.

(*b*) What maximum horsepower is required of the motor?

1·3. Beginning with water at 20° C how much would it cost for the electrical energy required to produce 200 lb of ice at 0° C assuming motor efficiency 80 per cent, compressor efficiency 60 per cent, and thermal losses due to imperfect insulation of water and refrigerant 50 per cent? Electrical energy is 2 cents per kwh.

1·4. A transmission line delivers 50 kw to a load made up of a number of motors which average 60 per cent efficiency. The power loss in the line is 500 watts. The transmission line is fed from a power plant of 70 per cent efficiency. Compute:

(*a*) The efficiency of the transmission line.

(*b*) The overall efficiency of the complete system.

1·5. Neglecting losses, at what horsepower rate may energy be obtained from a small waterfall 30 ft high, over which the flow of water is 500 cu ft per minute? (1 cu ft of water weighs 62.4 lb.) If the overall efficiency of this size hydro-plant is 25 per cent what kilowatt load can it handle?

1·6. It is desired to raise 1 pt (1.04 lb) of water from 20° C to 90° C in an electric water heater which takes 3 amp at 110 volts and has an efficiency of 80 per cent. Determine:

(*a*) The total electrical energy to be supplied from the line.
(*b*) The time rate at which this electrical energy is supplied.
(*c*) The time required to bring the water to 90° C.
(*d*) Cost of heating the water at 2 cents per kwh.

1·7. A manufacturer has a pump which, over its practical speed range, has an efficiency expressed by $\eta_1 = a(1 - kN)$ and requires power expressed by $P_1 = 3a/N$. An electric motor, which he intends to couple direct to the pump shaft, has an efficiency expressed by $\eta_2 = b\sqrt{N}$ over the same speed range and delivers power expressed by $P_2 = 2bN$. k, a, and b are parameters, and N equals speed.

(*a*) When $N = f(a,b,k)$ at what speed would the set be operated for maximum overall efficiency?
(*b*) What is the expression for this efficiency in terms of a, b, N?
(*c*) At what speed will the set operate?

1·8. In the modernizing of a candy factory it is desired to change to electric heating for the maintenance of vats of dipping chocolate at 50° C in a room which is at 29° C. When a vat with its capacity load of 60 lb of chocolate is allowed to cool from normal condition it is found to start cooling at the rate of 4.63° C per minute. It is estimated that 90 per cent of the electric heater output will reach the chocolate. The supply voltage is 220. Specific heat and specific gravity of the chocolate are 0.60 and 0.79 respectively. Compute the heater specification for one vat as follows:

(*a*) Kilowatt rating (input, of course).
(*b*) Ampere rating.
(*c*) Resistance (at operating temperature).

1·9. A certain industrial process requires 48 gal of water per minute at a temperature of 50° C. (1 gal of water weighs 8.33 lb.) The available water is at 20° C and must be raised 110 ft. As a 110-volt supply is available, it is decided to use a motor-driven, direct-connected pump with an electrical heating coil inserted in the discharge pipe of the pump.

For this load, the pump efficiency is 67 per cent, and the motor efficiency is 80 per cent. The power company's rate for electrical energy is 1.5 cents per kwh. Neglecting friction and heat losses in pipes and wiring, determine:

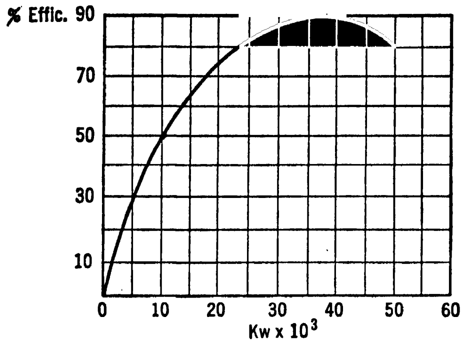
(*a*) Electrical power required to pump the water.
(*b*) Electrical power required to heat the water.
(*c*) Total cost of electrical energy per gallon of water.
(*d*) Percentage of total cost (*c*) for pumping alone.

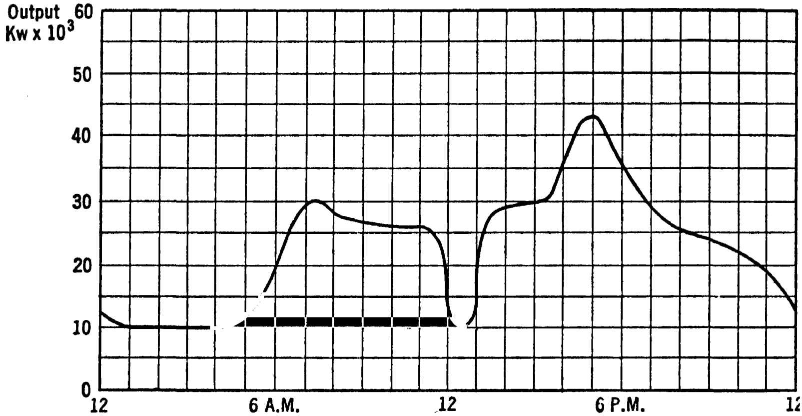
1·10. A copper water pipe of ½ in. inside diameter is at temperature −15° C. Thawing by electric current flow along the pipe is contemplated. The resistance of the pipe is approximately 5.6 microhms per foot of length and 50 per cent of the heat fails to reach the ice. Approximately how long will it take a current of 1000 amp just to melt $\frac{1}{10}$ of the volume of the ice? (Sp. ht. of ice = 0.504.)

1·11. A motor running 24 hours per day drives a load which requires

Full rated motor output 8 hrs. at 80 per cent efficiency
$\frac{1}{2}$ rated motor output 12 hrs. at 50 per cent efficiency
$\frac{1}{10}$ rated motor output 4 hrs. at 10 per cent efficiency

Compute the all-day efficiency of the motor.





1·12. Given the above data on the output and efficiency of a power plant.

(*a*) Draw the power input curve for the day and compute the all-day (energy) efficiency of the plant. Semigraphical solution is recommended.

(*b*) Determine by a general proof, or by actual computation of the above case, whether or not the average power efficiency for the day is the same as (*a*).

CHAPTER II

ELECTRIC CURRENT AND CONDUCTORS

2·1. Electrical Conduction. It is convenient to classify electrical conduction according to the three states of matter: gaseous, liquid, and solid. Our knowledge of electrical conduction in gases begins with the work of J. J. Thompson in the late nineteenth century and marks the beginning of what we may now call the *electronic era.* Here we find first evidence of the existence of the suspected "particle" of electricity, of its mass, of its charge, and of its size followed by a succession of theories of the structure of the atom as men labored to prove and disprove in a seeming near frenzy of attack concentrated even later on this one phase of physics. Largely as a consequence of this study of gaseous conduction we now conceive that all matter in all states is basically electronic.

The electron theory of the structure of matter is no longer a novel idea even to the nonscientific intelligence. That a student of electrical engineering should be entirely familiar with the elementary concept of the electron theory should require no emphasis; the sources of information are now numerous and range all the way from popularized newsstand articles to the most profound of technical and scientific publications.

Because conduction in gases and in liquids is quite generally introduced in the study of *electronics* and *general chemistry* respectively, we shall here turn immediately to the consideration of *solids only.*

2·2. The Electron Theory of Matter. The reader is familiar, presumably, with the concepts of the divisibility of matter into the ultimate particles of the molecular theory and of its further divisibility into the atoms of the chemist's atomic theory. It is indeed progress to feel assured that the almost innumerable variety of the substance of the universe can be accounted for by some mere ninety or so elements in various combination. To find that matter is still further divisible into even lesser particles of less than a half-dozen varieties seems like an approach to the millennium of man's understanding of his physical world. While the general idea of the electron theory which deals with this subatomic structure of matter is pretty well established there remains much to be done in clarifying the actual structure so that it will

fit or "explain" all the experimental evidence, and the millennium yet appears rather remote.

The electron theory was first conceived on the basis of two particles, the electron and proton. The simplest atom, hydrogen, comprises one ***electron*** of mass 1/1835 of the mass of the hydrogen atom and one ***proton*** with mass accounting for the other 1834/1835. Each particle possesses an equal charge or quantity of electricity of 1.602×10^{-19} coulomb, the electron negative and the proton positive. Atoms of other elements in general were conceived to possess a ***nucleus*** which comprised all the proton constituent and some of the equal numbers of electrons which constituted the normal uncharged atom.

The remaining electrons, external to the nucleus, were conceived by Bohr to pursue orbits around the nucleus as though for a miniature solar system. On this basis considerable progress ensued in "explaining" the various properties of matter. Eventually, however, the accumulation of inharmonious evidence proved too great to cope with, even for modifications of the theory. The matter is still unsolved but in the meantime notions of orbital electrons continue to serve usefully for many purposes. Probably the most staggering difficulty was introduced by the discovery that electrons may produce *interference* patterns similar to those of light which point to a wavelike aspect of the phenomenon and which launch for the electron the same old dilemma that has long beset theories of light, i.e., is it corpuscle or wave? The answer at present seems to be that it must be both, the one predominant over the other according to the particular circumstances!

In 1932 evidence was disclosed which indicated the existence of particles other than the electron and proton. A particle having essentially the mass of the proton but without electrical charge was termed the ***neutron.*** Another particle having essentially the mass of the electron but with positive charge was termed the ***positron.*** Whether the proton is possibly a close union of a neutron and a positron is not yet certain. For practical engineering purposes it does not yet seem necessary to go beyond the proton-electron concept of composition or Bohr's theory of a nucleus with orbital electronic structure but the future is likely to prove otherwise.

2·3. Electrical Conduction in Solids. According to the electron theory, the identity of all substances is determined by the structures of their nuclei. About these nuclei *extranuclear* electrons move in orbits more or less related to the nuclei. Some substances are characterized by associations between nuclei and extranuclear electrons so close and inviolate that the electrons are said to be ***bound electrons.*** Other substances are characterized by very loose relations between nuclei and

extranuclear electrons. These electrons are free to associate promiscuously with any other nuclei within the body of the substance or possibly other substances of like character which may be in contact with the first; they are called ***free electrons.*** Note however that the freedom of these electrons is not so complete as to admit of their readily departing from the confines of the body as entirely "free" electrons. When an electron does leave, it loses identity with any particular substance and is described by any of several names, depending on the process by which it escaped. Atomic bombardment, thermionic emission, and high-potential gradients are familiar causatives for electrons to leave the normal confines of the substance with which they have been associated.

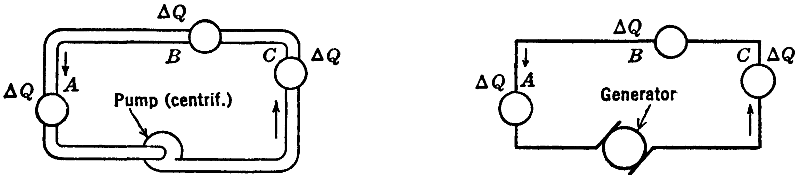


FIG. 2·1. Analogous hydraulic and electric circuits.

The substances characterized by their content of free electrons are called ***conductors.*** Substances characterized by a *lack* of free electrons are called ***insulators.*** The terms are merely relative opposites. A good insulator for one purpose may be a satisfactory conductor for another purpose and vice versa. The metals, as a group, easily comprise all the best conductors. Carbon and graphite follow in a secondary but uniquely important place.

When conductors are joined together in a closed loop, as when a wire is connected to the terminals of a generator, the free electrons progress around the loop as though doing a Maypole dance. Although at ordinary temperatures individual gymnastics are extremely vigorous and varied, their spacing, averaged over even extremely minute intervals, is so uniformly constant as to admit of picturing them for most purposes as rigidly maintaining exact and uniform spacing among themselves throughout all conductors at all observed temperatures.

This average rigidity or *incompressibility* of free electrons in a conductor greatly simplifies the analysis and computation of the electron displacement and the flow, or time rate of electron displacement, which we call ***current*** in electric circuits. It also validates analogs such as water flow in pipes and BB shot packed in a tube, which sometimes assist in visualizing the phenomena of electric circuits.

Such an analog is given in Fig. 2·1 for a generator with a conductor connected to its terminals. If along the path or "circuit" of the dis-

placement of water (or electrons) any number of stations such as A, B, C are set up for a *traffic count* of water molecules (or of electrons) the same count ΔQ will be observed at each station ($\Delta Q_A = \Delta Q_B = \Delta Q_C$ = etc.) during a given interval of time Δt. The same is true for an infinitesimal interval of time dt if absolutely no compressibility is admissible. Then *simultaneously at each and every station* the same rate of flow or *current* will be observed

$$i = \frac{dq}{dt}$$

The important consequence is evident; one observation or measure alone suffices to provide data about the current throughout the single path circuit of electron displacement.

As before indicated, the accepted physical picture of current actually is not so simple as this which suffices for ordinary engineering use. The free electrons are pursuing their assorted paths, like molecules in a confined gas, with much zeal even when no *current* exists. Their average velocity at ordinary temperatures is considered to be of the order of 10^8 centimeters per second (40,000 miles per minute)!

Now let us take a copper conductor of 1-sq cm section and instead of a normal current of 100 amp, let us consider an extremely *heavy* current of 10,000 amp (coulombs per second) which is sufficient to melt the conductor in less than a minute! Since the electron charge is 1.602×10^{-19} coulomb, this current requires a flow or *traffic count* of $10{,}000/(1.602 \times 10^{-19}) = 6.24 \times 10^{22}$ electrons per second along the conductor. The number of free electrons per atom of copper is not certain but may be taken as one per atom. We can compute the number of atoms per cubic centimeter of copper if given Avogadro's number $N_0 = 6.061 \times 10^{23}$ molecules per gram-mole, the atomic weight of copper 63.57, and the density of copper 8.89 grams per cubic centimeter. The number of atoms per cubic centimeter of Cu will be

$$\frac{8.89}{63.57} \times 6.06 \times 10^{23} = 8.47 \times 10^{22}$$

From our assumption of one free electron per atom it follows that Cu has 8.47×10^{22} free electrons per cubic centimeter. With this number of electrons *participating* in the displacement, the current will require an average velocity of electrons along the conductor

$$v = \frac{6.24 \times 10^{22}}{8.47 \times 10^{22}} = 0.74 \text{ cm per sec}$$

or only $\frac{1}{60}$ mile per hour. It should now be clear that to describe even an extremely heavy electric current as an electron *drift* is altogether appropriate, and that the conduction current has little in common with the electron speed of orbital motion. Possibly it seems difficult to reconcile this fact with the popular conception that conduction of electricity is the epitome of speed. The speed comes about by reason of the previously noted *incompressibility* characteristic of metallic conduction. By analogy, if a rigid pipe is filled with incompressible fluid any displacement of fluid at one end is immediately duplicated all along the pipe and at the opposite end. In the same sense that the pipe experiment may fail because the elasticity of the pipe may seriously violate the stipulated use of *rigid pipe,* so may the electrical experiment fail because the conductor boundaries, collaborating with the surrounding dielectric or *insulating* medium, may exhibit excessive capacitive or "elastic" effects. These effects are by no means unimportant and will be discussed in later chapters. They in no way conflict with the incompressibility concept here presented.

2·4. Ohm's Law. Perhaps the most important—certainly the most advertized—practical relation concerning electrical conductors is expressed in Ohm's law. ***The potential difference between any two points of a simple current-carrying conductor is directly proportional to the current between the points and to the resistance of the current path between the points.***

Note here that the term *conductor* has been used rather than the more usual term *circuit.* This distinction is important if we are to keep the basic facts clear. To trace the present ambiguity of statement it is necessary to observe that it was in 1827 that Dr. G. S. Ohm published his work on "The Galvanic Circuit Investigated Mathematically." Ohm's "circuit" consisted of a length of copper *wire* and a *battery.* Today we use the term circuit to include all manner of connections and interconnections of electrical equipment such as were not even dreamed of in 1827. Like many other words circuit has changed in meaning to fit the requirements of the age and we must appreciate that the term *conductor* more closely translates Ohm's "circuit" into twentieth century parlance.

It is of interest that Ohm enunciated no specific law—he simply stated his deductions regarding the behavior of electric "circuits" by recognizing and exploiting an analogy between electrical and thermal conductivity. Although Ohm in 1826 had published the experimental evidence which prefaced his theoretical exploitation of Fourier's study of heat flow (1822) his work was not well received. It was not until Wheatstone's work in 1853 that the experimental proof of Ohm's theory

was accepted widely enough to elevate it to the realm of law as we now know it.

2·5. Resistance and Resistivity. Ohm's law is symbolized by $R = E/I$. This may be called a *functional* relation because it characterizes the performance or *action* of resistance R in determining for a live circuit the relation of the voltage E across R to the current I within it. Ohm's law gives no indication of the physical constitution of resistance or of the composition and construction of a resistor. Nevertheless resistance is a physical quantity which, with little exception, is independent of the existence of current or voltage and depends only on the ***composition, dimensions,*** and ***temperature*** of the resistor.

Sir Humphry Davy, of safety lamp fame, observed what we may term the "constructional" aspect of resistance in 1821, six years before Ohm's famous work, and expressed his discovery in terms now symbolized by

$$R = \rho \frac{l}{A}$$

where l is the length of a resistor of uniform cross section A and of uniform composition expressed by ρ. Factor ρ is called the ***resistivity*** (sometimes *volume* resistivity) and is simply the resistance of a unit-dimensioned ($l = 1$, $A = 1$) volume of the given material at a given uniform and fixed temperature. It is customary to designate the temperature in degrees centigrade as a subscript of ρ. Values ρ_0 at 0° C and ρ_{20} at 20° C are available in tables for the resistivity of all metals and for some other materials.

Copper, of course, occupies a place of prime importance among conductors. Its resistivity varies considerably as a function of the nature and degree of chemical impurity and of the metallographic or crystalline structure. To facilitate the unambiguous specification of copper for electrical purposes the ***international annealed copper standard*** was established in 1913 by the International Electrotechnical Commission. The standard is conveniently expressed in terms of resistivity at 20° C.

$$\rho_{20} = 1.7241 \times 10^{-6} \text{ ohm-cm}$$

Copper of other quality is conveniently specified in terms of its *conductivity* (reciprocal of resistivity) expressed in per cent of the conductivity of the standard. Copper of greater than 100 per cent conductivity of course is possible because only a practical degree of commercial purity rather than an ideal maximum is represented by the standard.

It is convenient to think of resistivity as represented by the resistance of a unit-dimensioned cube. On this basis it is not unusual to find resistivity given in *ohms per centimeter cube* instead of *ohm-centimeters.*

This has the disadvantage of being susceptible to confusion with *ohms per cubic centimeter* which is by no means the same thing. Resistance is *not* basically a function of volume. A 1-ohm resistor, for example, may be of any desired volume, the choice depending largely upon the rate at which heat must be dissipated from it to avoid excessive temperature in any specific case.

The resistivity of a material is most readily converted into other units of dimension by using Davy's law $R = \rho(l/A)$ to find the resistance of the new unit in terms of the known units. For standard copper in the centimeter units given above, we convert to inch units as follows, using 1 in. = 2.54 cm.

$$R = \rho_{20}\frac{l}{A} = 1.7241 \times 10^{-6}\frac{2.54}{(2.54)^2} = 0.6788 \times 10^{-6} \text{ ohm}$$

and the new unit of resistivity, therefore, is

$$\rho'_{20} = 0.6788 \times 10^{-6} \text{ ohm-in.}$$

2·6. The Circular Mil Foot. Resistivities which involve the conventional *square* units of area are unwieldy for the sectional area of wires of *circular* section. It is convenient to establish for these a new unit of area which is circular and small enough to avoid subunit values for commercial wire sizes. This unit is the circular mil. ***The circular mil is a unit of area equal to the area of a circle of 1 mil (1 milli-inch) diameter.***

While it is true that we cannot visualize, as in square measure, the actual fitting of these unit circles into an unknown area as a mechanical process, the application of the unit becomes apparent in terms of the familiar geometric theorem that the areas of similar figures are related as the squares of any corresponding linear dimension. Thus we find that: ***The area of any circle in circular mils is readily expressed as the square of its diameter in mils or thousandths of an inch.***

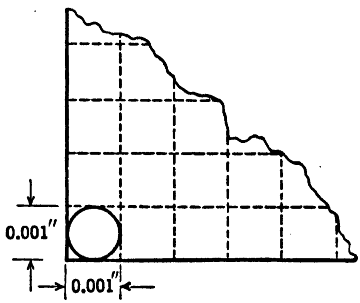


FIG. 2·2. The circular mil unit of area.

By this means the $\pi/4$ factor in the $A = (\pi/4)d^2$ relation of square measure is avoided. Conversion from one unit to the other, however, is necessary and, for this, it is safer to visualize the basis for conversion than to depend on rote memory for the conversion factor. For this let us divide the inch square into mil squares as indicated in Fig. 2·2.

There will be $(1000)^2 = 10^6$ of these mil squares in the inch square. The circle inscribed in one of these squares has an area of 1 circular mil which obviously is smaller than the square according to the factor $\pi/4$. Since the number of mil circles equivalent to the inch square area must be greater than the number of mil squares, we reason that *both* factors work to make the number of circular mils *greater* and write

$$1 \text{ sq in.} = \frac{4}{\pi} \times 10^6 \text{ cir mils}$$

or

$$\text{Area in square inches} \times \frac{4}{\pi} \times 10^6 = \text{Area in circular mils}$$

Because the foot is more convenient than the inch for wire lengths, the wire unit is taken of this length and is termed a ***circular mil foot.*** Thus it is conveniently visualized as a cylinder of dimension $l = 12$ in. and $A = \frac{\pi}{4} \times 10^{-6}$ sq in. Substituting in Davy's law we find for standard copper at 20° C

$$R = \rho_{20}\frac{l}{A} = 0.6788 \times 10^{-6}\frac{12}{(\pi/4) \times 10^{-6}} = 10.371 \text{ ohms}$$

and the new unit of resistivity is, therefore,

$$\rho'_{20} = 10.371 \text{ cir mil ohms per ft}$$

In this case the true dimension *circular mil ohms per foot,* corresponding to the ohm-inch, is not as commonly used as the ohms-per-inch-cube type of designation *ohms per circular mil foot.* Unfortunately this erroneously implies ohms per unit volume and the initiate must guard against it; *circular mil foot does* ***not*** *mean volume.*

2·7. Effect of Temperature. The electrical resistance of all materials is more or less of a function of temperature. For most pure metals the function is reasonably linear over the usual operating range and increases with increase in temperature. A typical plot of experimental data on a copper conductor immersed in an oil bath at different temperatures is given in Fig. 2·3.

For most metals even small amounts of impurities markedly affect the resistance-temperature characteristic to the extent that it is useful as a sensitive test of the purity of the given metal. In general impurities reduce the variation of resistance with temperature, but the exact

amount and law of the effect in a particular case is indeterminate except by experimental study. This is equally true of alloys although sufficient experimental work has been done to catalog the behavior of those commonly used.

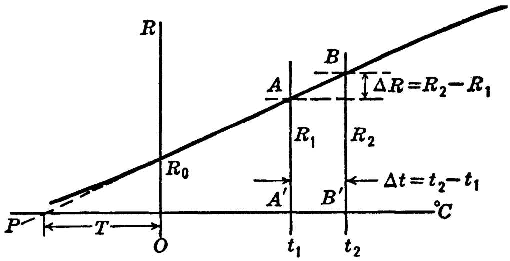


FIG. 2·3. A typical resistance-temperature characteristic for copper.

2·8. The α-Concept. For the linear resistance-temperature function of Fig. 2·3, a resistance change $\Delta R = (R_2 - R_1)$ is directly proportional to the corresponding temperature change $\Delta t = (t_2 - t_1)$. The physicist expresses this relation in terms of a factor α which represents the fractional change in resistance $(\Delta R/R)$ per degree of temperature change (Δt). For the case in Fig. 2·3 the fractional rise in resistance is

$$\frac{R_2 - R_1}{R_1} = \alpha_1(t_2 - t_1) \qquad [2\cdot1]$$

The factor α_1 is known as the ***temperature coefficient of electrical resistance.*** Note that the left term of equation 2·1 is dimensionless. The right term, of course, must likewise be dimensionless.

It follows that α must have simply the dimension of *reciprocal temperature,* as previously indicated by stating that it is *fractional increase per degree.* Let it be clear that *α does **not** in any way contain the dimension resistance.* While reciprocal degrees centigrade are usual, conversion to the Fahrenheit scale is readily performed if desired.

Note especially that α is not a constant but varies with the reference or base value R_1. This is best indicated by rewriting equation 2·1 to give

$$\frac{R_2 - R_1}{t_2 - t_1} = R_1\alpha_1 \qquad [2\cdot2]$$

The quantity $R_1\alpha_1$ represents the slope of the R-t graph which, for the linear case, is constant and requires that α_1 vary inversely with R_1. It is usual to subscript α_1 (and sometimes R_1) with the temperature of R_1, e.g., for $t = 20$: α_{20} and R_{20}.

Values of α at different temperatures, usually α_{20} at 20° C, are readily available from numerous handbook and other data sources for most materials of commercial interest.

The relation given by equation 2·1 or 2·2 is commonly arranged for solution of an unknown value R_2 and is familiar to physics in this form:

$$R_2 = R_1[1 + \alpha_1(t_2 - t_1)] \qquad [2\cdot3]$$

2·9. The *T*-Concept. It is not unusual that the known value of α is neither for t_1 nor t_2. Because of this not insurmountable but annoying complication, it is the practice in engineering to set up for the linear case in Fig. 2·3 a simpler relation which avoids pursuit of the elusive α.

This is best derived by extending the plotted straight line of Fig. 2·3 to intersection P with the temperature axis. Although point P is beyond the realm of experiment and has absolutely no factual significance or relation to absolute zero, it provides similar triangles PAA' and PBB' which give the following proportion.

$$\frac{R_2}{R_1} = \frac{T + t_2}{T + t_1} \qquad [2\cdot4]$$

The quantity T as seen in Fig. 2·3 is, unlike α, a dimension of the graph and a *true constant* throughout the range of linearity. Knowing the value of T, equation 2·4 is readily handled on the slide rule for solution of any one unknown. Although values of T are not directly available from handbook sources, they are easily computed from the available values of α. The relation is readily derived by equating (eliminating) R_2/R_1 from equations 2·4 and 2·3 to give

$$1 + \alpha_1(t_2 - t_1) = \frac{T + t_2}{T + t_1}$$

$$\alpha_1(t_2 - t_1) = \frac{T + t_2}{T + t_1} - 1 = \frac{T + t_2 - T - t_1}{T + t_1}$$

$$= \frac{t_2 - t_1}{T + t_1}$$

or

$$\alpha_1 = \frac{1}{T + t_1} \qquad [2\cdot5]$$

and

$$T = \frac{1}{\alpha_1} - t_1 \qquad [2\cdot6]$$

For the usual case where α_{20} for $t = 20°$ C is given,

$$T = \frac{1}{\alpha_{20}} - 20 \qquad [2\cdot7]$$

When α_0 for $t = 0°$ C is given, equation 2·6 becomes simply

$$T = \frac{1}{\alpha_0} \qquad [2\cdot8]$$

The relation of α_1 to temperature t_1 is readily graphed from equation 2·5 as shown in Fig. 2·4. Because T is constant, $\alpha = f(t)$ is a hyperbola

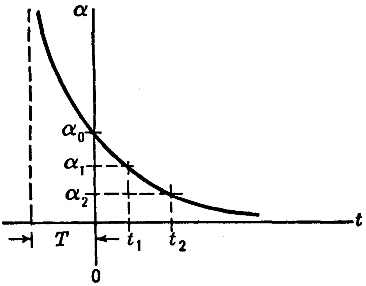


FIG. 2·4. Showing the variation of temperature coefficient of electrical resistance α with temperature t.

and clearly indicates the objectionable elusive character of the α value in contrast with the constant T quantity.

For ***copper*** the value of T is 234.5° C and is easy to remember. Values for some other materials are listed in the table below.

Material	ρ_{20} Resistivity microhm-cm 20° C	α_{20} Temperature coefficient 20° C	$\beta_{20} \times 10^{-6}$	$T = \frac{1}{\alpha_0}$
Aluminum (Amer. std.)	2.67	0.00403	1.8	228
Carbon	5000±	−0.0003±	......	−3,300±
Constantan	49	±0.000008	......	±160,000
Copper, annealed	1.724	0.00393	0.43	234.5
Nickel (pure)	7.2	0.0061	6.0	1,100
Nichrome	110	0.00015	......	6,650
Platinum	10.0	0.0031	−0.6	302
Silver (99.98% pure)	1.64	0.0038	......	243

2·10. A Simple Relation for Copper. It is entirely feasible to write equation 2·3 for α_0 in the form

$$R_t = R_0 + R_0\alpha_0 t \qquad [2\cdot 9]$$

Neglecting the effect of temperature on the dimensions of a material, we may write equation 2·9 in terms of resistivities ρ_0 at 0° C and ρ_t at t° C.

$$\rho_t = \rho_0 + (\rho_0\alpha_0)t \qquad [2\cdot 10]$$

For most metals the error incurred for temperatures between 0° and 100° C is negligible. Nevertheless, it is not usual to find the known data for materials tabulated in the simple form of resistivity ρ_0 and a resistivity-temperature coefficient expressed by $\rho_0\alpha_0$ as suggested by equation 2·10. Although this form is not to be advocated in place of the usual as a general-purpose form, it is sufficiently advantageous in the much used copper computations to warrant setting up equation 2·10 for at least this particular material in terms of the circular mil foot resistivity.

$$\rho_t = 9.55 + 0.0408t \qquad [2\cdot 11]$$

It is worth noting from this equation that the resistivity of copper in ohms for the circular mil foot changes 0.0408 ohm per degree centigrade or nearly 4.1 ohms over the 0–100° C range. This is for *copper* of 100 per cent conductivity as discussed earlier in this chapter.

2·11. Nonlinear *R-t* Characteristics. When the nature of the material or the temperature range is such that serious departure from the linear *R-t* function occurs, it becomes necessary to do one of two things:

1. To use a different value of T for each reasonably linear section of the *R-t* graph.

2. To use a second-order (and rarely higher orders of) temperature coefficient. Tables are available not only for $\alpha_1 = 1/(T + t_1)$ but also sometimes for β coefficients which extend equation 2·3 to the form

$$R_2 = R_1[1 + \alpha_1(t_2 - t_1) + \beta_1(t_2 - t_1)^2] \qquad [2\cdot 12]$$

A few β values are included in the table on page 27. Of the common metals it should be noted that nickel has the largest β coefficient and therefore the greatest departure from a straight line.

Many *alloys* are *far from linear* over their entire possible temperature range. So-called ***zero-temperature coefficient alloys*** such as the Cu-Mn-Ni alloy commercially known by the trade name Manganin have *R-t* characteristics with the temperature coefficient positive at low temperatures and negative at high temperatures. The actual zero-temperature coefficient, by composition control, is made to fall in the desired operating region and is sufficiently small (±) over a sufficient

temperature range to solve the problem for instrument resistors which usually can be permitted to vary but a fraction of 1 per cent over the reasonable temperature range encountered by the instrument during operation.

2·12. Resistance Thermometry. Aside from the obvious use of the foregoing to compute the consequence of a temperature variation in the resistance of electric circuits for various kinds of electrical equipment, it is important to consider the converse utilization of the phenomenon to measure temperature by measurement of electrical resistance. This is done in either of two ways:

1. By measuring the resistance of electrical conductors in equipment such as transformers and dynamos first at room temperature and then after operation. This is a standard test procedure for electrical machinery and readily provides the average temperature or temperature rise of the conductors or windings by simple measurements and computation.

In addition to its application to the regular operating windings, large generators are commonly provided with small coils built into their windings for the sole purpose of temperature measurement by this means. Their resistance is usually measured with a Wheatstone bridge.

2. By constructing a ***resistance thermometer*** comprising a small coil of wire in a protective tube of porcelain or other noncontaminating enclosure or "bulb." Platinum wire of high purity is well recognized to be pre-eminent for this purpose and enables the attainment of accuracy as high as one part in 10,000. For a less expensive thermometer nickel is commonly used with but little sacrifice in accuracy or range and considerable gain in sensitivity. Nickel, however, in contrast to platinum and all other pure metals, departs notably from a linear R-t characteristic and must be dealt with in full recognition of this fact.

These instruments, when properly constructed, readily exceed the accuracy of the best glass thermometers of the liquid expansion type and are especially noted for their sensitivity, extensive range, and constancy of calibration. Resistance thermometers are particularly adapted to measuring extremely low and moderately high temperatures, especially temperature *differences.* They are practical up to at least 1000° C.

For industrial use, the resistance thermometer finds notable competition from the *thermocouple.* The latter is preferred for many applications and the choice of one over the other is often a matter of preference or economics. Where spot temperature is desired or space is at a premium, the smaller size of the thermocouple is advantageous.

2·13. Wire Gages. The use of gage numbers for identifying the sectional sizes of wires may be attributed to the nature of the drawing

process in manufacture. Wire is reduced to successive sizes by *drawing* or pulling through dies of progressively smaller size. Depending on the material, there is a maximum practical reduction possible in each die. It is natural to number these dies 1, 2, 3, etc., and to apply these same numbers to the size of wire pulled from each die.

Because of difference in both materials and skill of manufacture many diverse wire gages have been in accepted use and others have been proposed. In America two have survived for extensive use:

(*a*) American Wire Gage (AWG) also known as Brown and Sharpe (B&S) J. R. Brown, 1857.

(*b*) Steel Wire Gage (SWG).

In England one wire gage, the Standard Wire Gage, is legal.

In Germany, France, Austria, Italy, and other continental countries wire gages are little used, the diameter in millimeters being specified directly—sometimes the millimeter wire gage. There is some tendency in this country toward the use of diameters in decimals of an inch but gage numbers and circular mil specifications may be expected to predominate indefinitely.

At present, the AWG or B&S gage is so commonly used that knowledge of it is imperative for practical work with copper and aluminum wires and cables. Although tables are readily available, it is worth while to recognize the basic plan or pattern of these tables and to memorize certain "landmarks" or values which will translate gage numbers into a sense of actual sizes.

The B&S gage is specified by the diameters of two gage numbers and a law relating the intervening sizes as follows:

1. No. 0000, diameter 0.4600 in.
2. No. 36, diameter 0.0050 in.
3. The 39 intermediate steps are in geometric progression.

This last means that the size of any pair of adjacent gage numbers, n and $n - 1$, have the same ratio as that of any other pair of adjacent gage numbers. The ratio x is easily computed for the diameters of adjacent gage numbers, remembering that *size* goes *up* as gage *number* goes *down*.

$$x = \frac{D_{n-1}}{D_n} = \sqrt[39]{\frac{0.4600}{0.0050}} = \sqrt[39]{92} = 1.123 \qquad [2 \cdot 13]$$

Because resistance depends on *area* rather than diameter, the ratio of areas is significant. It is readily found from the ratio of adjacent diameters.

$$\frac{A_{n-1}}{A_n} = \left(\frac{D_{n-1}}{D_n}\right)^2 = X^2 = (1.123)^2 = 1.261 \qquad [2 \cdot 14]$$

Note further that

$$\frac{A_{n-2}}{A_n} = \left(\frac{A_{n-1}}{A_n}\right)^2 = X^4 = 1.59 \qquad [2\cdot15]$$

and that

$$\frac{A_{n-3}}{A_n} = X^6 = 2.0050 \qquad [2\cdot16]$$

This last provides the convenient approximation that ***area doubles* (*or halves*) *for every three gage intervals.***

It is also useful to note that the ratio of areas for every *ten* gage intervals (n to $n - 10$) is approximately *ten*. The exact value of the ratio is

$$\frac{A_{n-10}}{A_n} = (X^2)^{10} = 10.164 \qquad [2\cdot17]$$

Unfortunately this approximation introduces twice as much error as does the three interval approximation first discussed, so that even for ten or more gage intervals, the doubling approximation is preferable. This is readily established by computing the per cent error *per gage interval* for each of the two approximations, as follows.

For the three interval ratio, equation 2·16 shows that the true value exceeds approximate value 2 by 0.0050 so that 2 is in error by 0.0050/2 = 0.0025 or 1/4 per cent. Since this is for three intervals the error per gage interval is one-third of this or 1/12 per cent.

For the ten interval ratio, equation 2·17 shows that the true value exceeds approximate value 10 by 0.164 so that 10 is in error by 0.164/10 = 0.0164 or 1.64 per cent. The error per gage interval, of course, is then 0.164 per cent which is very nearly 1/6 per cent or twice the 1/12 per cent incurred by the three interval ratio.

With the multipliers 2 and 10 in mind, it remains only to provide actual data for some one gage number. This is most judiciously taken for ***No. 10 copper wire*** of 100 per cent conductivity at 20° C.

R = 1 ohm per 1000 ft
D = 0.1 in.
A = 10,000 cir mils
M = 31.4 lb per 1000 ft (note 31.4 = 10π approx)

In dealing with resistance it must be clear that the ***resistance** ratio is the inverse of the area ratio* so that for equation 2·16:

$$\frac{R_{n+3}}{R_n} = X^6 = 2.0050 \qquad [2\cdot18]$$

It is frequently useful to note that ***aluminum*** has about 60 per cent of the conductivity of copper and about 30 per cent of the specific gravity.

With the above in mind we are equipped to estimate quickly and closely the *size, weight,* and *resistance* of any length and AWG size of copper or aluminum wire.

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QUESTIONS

2·1. Outline the basic features of the electron theory.

2·2. On the basis of the electron theory, distinguish between conductors and insulators.

2·3. What is our present-day concept of the electron movement which we call an electric current in a conductor?

2·4. Expound the distinction between electric conductor and electric circuit.

2·5. State Ohm's law *accurately* and concisely. State and explain the precautions to be observed when applying Ohm's law.

2·6. From what analogous study did Ohm conceive his law? Expound the analog *fully*.

2·7. Is it possible for a given current flow, say 100 amp, in a simple series circuit to have different average velocities of electron flow? Explain fully and illustrate.

2·8. What data are required to determine *completely* the resistance of a body and how are these data organized, i.e., in handbooks, for ready application?

2·9. Expound the meaning and use of T pertaining to change of resistance with temperature.

2·10. Given α_1, the temperature coefficient of electrical resistance at temperature t_1, derive an expression for α_2, the temperature coefficient at temperature t_2, in terms of α_1, t_1, and t_2.

2·11. For a fixed mass of copper wire at fixed temperature, derive a relation giving the electrical resistance as a function of its length.

2·12. (*a*) How did wire gages originate?

(*b*) What are the basic features of the B&S gage (not of No. 10 wire)?

2·13. Define circular mil and explain concisely *how* and *why* it is used.

2·14. Give the major physical properties of No. 10 copper wire as they concern the electrician.

2·15. Explain why ρ is ***not*** given in ohms per cubic centimeter.

2·16. Compute the *exact* value of A_{n-10}/A_n to five significant figures.

PROBLEMS

2·1. Compute the velocity of electron drift in a silver conductor carrying 10,000 amp per sq cm, assuming an average of one free electron per atom.

2·2. (*a*) Compute the rate of heat production in Btu per minute for one foot length of the above silver conductor carrying the 10,000 amp through the square centimeter section.

(*b*) Repeat the computation for the same size copper conductor carrying the same current and compare with (*a*).

2·3. No. 14 AWG copper wire (the smallest allowed in house wiring) has a diameter of 0.064 in.

(*a*) *Compute* its area in circular mils.

(*b*) *Compute* the resistance per 1000 ft at 25° C.

2·4. A block of annealed copper has the dimensions $X = 10$ in., $Y = 0.5$ in., and $Z = 0.2$ in. (volume $XYZ = 1$ cu in.). Determine the resistances R_1, R_2, and R_3 at 20° C (between respective pairs of opposite faces of the block).

2·5. If 1000 ft of No. 8 AWG copper wire are drawn through a die down to No. 14 size, determine by the approximate rule the length, weight, and total resistance of the new piece of wire. (Assume the properties of the material to remain unchanged.)

2·6. A copper bus bar of 10-ft length is to have a resistance of 0.0001 ohm at 50° C.

(*a*) Determine the circular mils cross section if a solid cylindrical conductor were used.

(*b*) What is the diameter of the conductor of (*a*) in inches?

(*c*) How many square inches of rectangular section would be necessary?

2·7. The four field coils (copper wire) of a motor have a resistance of 10 ohms each at a room temperature of 40° C. They are connected in series.

(*a*) After being in operation for several hours the total resistance has increased to 47.2 ohms. What average temperature rise per coil is indicated?

(*b*) If each field coil is wound with 1000 ft of copper wire, determine the diameter in ***inches*** of the wire used.

2·8. The diameter of wire for many tungsten lamp filaments is checked during the drawing process by weighing an accurately measured length.

(*a*) Compute the weight of a 40-watt, 115-volt, filament which is 14.5 in. long and 0.0013 in. diameter.

(*b*) Compute the resistivity, in circular mil ohms per foot, of the filament at operating temperature.

(*c*) Compute the diameter of a 100-watt, 230-volt, filament which has the same resistivity as the above 40-watt lamp but a length of 35.1 in.

2·9. The resistance of the lamp filaments in problem 2·8 at operating temperature 4450° F is 14.2 times their resistance at 70° F.

(*a*) Assuming a linear resistance-temperature relation, compute the value of $T = 1/\alpha_0$.

(*b*) A value of α_0 commonly given for tungsten in tables of physical properties is 0.0047. Explain why your value does not check this.

2·10. For a 25-mile transmission line with total resistance 4 ohms, it is desired to compare aluminum and copper. The conductivity of aluminum is 61 per cent that of copper. The weights of aluminum and copper are respectively 0.097 and 0.32 lb per cu in. Let the cost of aluminum and copper be taken at 55 and 30 cents per lb respectively. Compute, for the lines, the *ratio* (aluminum to copper) of their:

(*a*) Diameter.

(*b*) Weight.

(*c*) Cost.

Chapter III

ELECTRICAL MEASURING INSTRUMENTS

3·1. Electrical Measure. It has been truly said that knowledge in any field is of a meager and unsatisfactory kind until you can express it quantitatively. That this is especially true of engineering requires no argument. The electrical engineer is particularly blessed with exceptional facilities for the precise measurement of many of the quantities which concern him. So fine are his instruments that they are now commonly being adapted to the measurement of many quantities which basically are not electrical. Among these may be cited time, temperature, force, length, and speed. Electrical instruments are now finding application in an ever growing array of fields outside of engineering; the psychologist, physiologist, biological and clinical physician, to mention a few, are all improving their work by aid of electrical measuring instruments.

The scope of this study permits only an introductory survey of so large a subject. We shall confine our consideration to the major category of instruments which are of the direct indicating and recording types as contrasted with those (potentiometers, bridges, etc.) which require an operator. Indicating instruments commonly employ a pivoted pointer, light beam, or a cathode ray actuated by forces obtained from one or more of the basic sources already familiar from physics as follows.

1. *Electrostatic*
   a. Repulsion of like electrostatic charges.
   b. Attraction of unlike electrostatic charges.
   c. Tendency of components of an electrostatic circuit to orient themselves for maximum capacitance or maximum electrostatic flux.
2. *Magnetic*
   a. Repulsion of like magnetic poles.
   b. Attraction of unlike magnetic poles.
   c. Tendency of components of a magnetic circuit to orient themselves for maximum permeance or maximum magnetic flux.
3. *Electromagnetic.* Force on an electron current (or current-carrying conductor) in and perpendicular to a magnetic field.

4. *Electrothermal.* Thermal expansion accompanying the heating of suitable current-carrying materials.

3·2. Electromagnetic Force. In 1820 Oersted discovered that a compass needle is deflected by a nearby wire carrying current in the direction of the needle. This marks the beginning of our knowledge of electromechanical phenomena. In modern parlance it is an experimentally determined fact that mechanical force is exerted on electrons which are moving in a magnetic field with a component of velocity

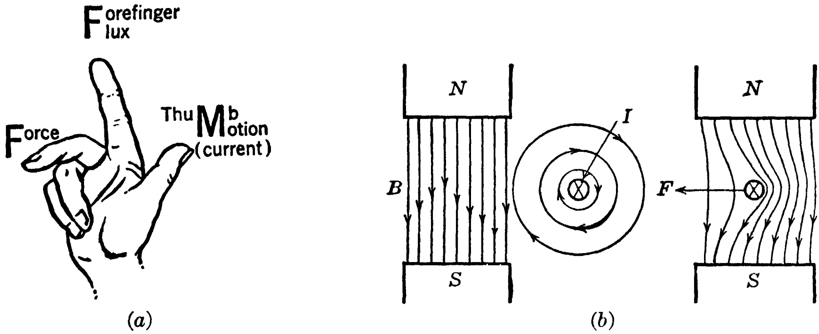

FIG. 3·1. The direction of force from an electric current in and normal to a magnetic field.

perpendicular to the field. The force, velocity, and field are mutually perpendicular as represented in Fig. 3·1.

These directions are best represented in the notation of vector algebra. In lieu of this some kind of rule of thumb is required. Two of the most commonly used are as follows.

1. Place thumb and adjacent two fingers of the *right* hand mutually perpendicular, the middle finger perpendicular to the palm (Fig. 3·1*a*). When the thumb indicates direction of conventional current, and the adjacent finger the flux, the middle finger indicates direction of force exerted by the conductor.

2. Sketch the direction of the flux which would be produced by the current alone (Fig. 3·1*b*). Combine this with the flux already identified as of density B. If the resulting flux lines are imagined to be stretched elastic bands, the implied direction of force is correct.

Note that *conventional* current rather than *electron* current is referred to in each rule. This unfortunate conflict between the old and the new representation of electric current seems destined to annoy even the future generation but need occasion no confusion if the designations "conventional" and "electron" are prudently employed.

The magnitude of the force is found to be directly proportional to the density B of the field at the electrons, to the transverse velocity v of the electrons, and to the electron charge dq as follows.

$$dF = Bv\,dq \tag{3·1}$$

F = newtons (one newton = 0.2248 lb).
B = webers per square meter (one weber = 10^8 lines or maxwells).
v = meters per second.
q = coulombs.

When the electrons are moving in a conductor of length l wholly within and normal to the field of uniform density B, the integrated force produced by the conductor may be expressed in terms of the current $i = dq/dt$ as follows.

$$dF = B\frac{dl}{dt}\,i\,dt = B\,i\,dl$$

$$F = \int B\,i\,dl \tag{3·2}$$

Since flux density B and current i are both taken to be uniform and constant throughout the length l of the conductor, the integral reduces to

$$F = Bli \tag{3·3}$$

l = meters.
i = amperes, or coulombs per second.
F and B units as before.

3·3. Oscillographs. It commonly occurs that the electrical quantities which concern the engineer are far from constant. Precise knowledge of how these vary with time is often important, even to microsecond intervals.

Instruments have been available for some years which will provide a visual or photographic graph of two coordinates actuated more or less directly by the phenomenon itself. Although several types of these oscilloscopes or oscillographs have been devised, two have emerged pre-eminent, the Duddell or moving-coil and the cathode ray.

The moving-coil oscillograph is mechanically actuated by electric current and has been designed to follow variations accurately up to only a few thousand cycles per second. The cathode-ray oscillograph is devoid of mechanically moving parts and is readily designed to follow variations of 10^6 and more per second. While the cathode-ray oscillograph is nearly as old as the discovery of cathode rays it is only since

the first quarter of the century that it has emerged from the research laboratory as a thoroughly practical engineering instrument of diverse application.

These instruments are now of such general importance in the several branches of science as well as engineering, and are so helpful in the study of electrical phenomena that we shall at once consider them in some detail.

3·4. The Moving-Coil Oscillograph. The moving-coil or Duddell oscillograph utilizes the electromagnetic force principle of Oersted just described and symbolized by $F = Bli$. A schematic diagram is given

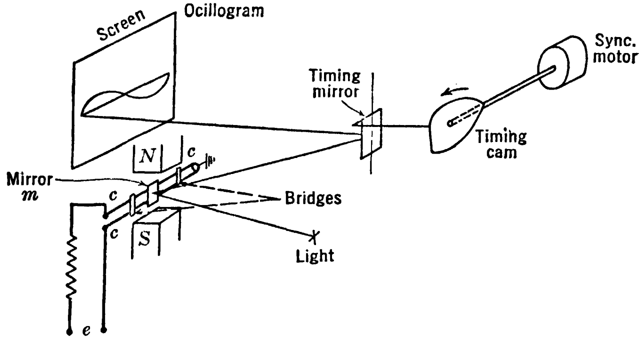

FIG. 3·2. Schematic of Duddell oscillograph.

in Fig. 3·2. The magnetic field from poles NS is made constant and the control current I_y is conducted directly through the field so that $F_y = BlI_y$. The ***conductor element*** comprises a hairpin-shaped, one-turn coil (ccc) of very small phosphor-bronze ribbon supported in tension on tiny bridges as though for a stringed musical instrument. Since the electron flow in one coil side is opposite to that in the other, the forces are opposite and comprise a torque on a tiny mirror (m) cemented to the midpoint of the coil sides.

A slender ***beam of light*** is directed onto mirror (m) then reflected to a timing mirror which rotates on an axis perpendicular to that of the first mirror, and thence to a translucent screen. The "coil" or vibrator element is enclosed in a container provided with a window and filled with oil as a damping medium.

The ***timing mirror*** in some models is deflected by a cam so as to move the light beam along the screen in proportion to time and quickly return it at the completion of the cycle (or cycles) of the periodic current I_y being investigated. A low-power synchronous motor is used in addition to the hand-controlled main motor to hold the cam speed in synchronism with I_y.

In competent hands this oscillograph is a very satisfactory instrument for low-frequency phenomena. However the mirror and coil assembly or ***vibrator*** will stand but little overload current without injury and their repair requires appropriate and experienced technique. With few exceptions, these instruments are not readily portable and cost in excess of $1000.

3·5. The Cathode-Ray Oscillograph. While an understanding of the complete cathode-ray oscilloscope is beyond the purpose of this study

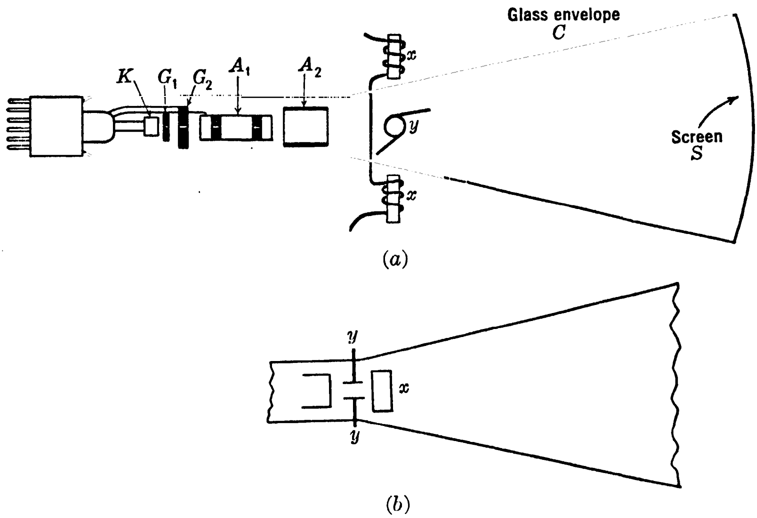

FIG. 3·3. A cathode-ray tube and its parts for (a) magnetic and (b) electrostatic deflection.

the essential elements are readily indicated and special attention to the phenomena by which the cathode-ray beam is deflected or controlled as a "pointer," like the light beam of the moving-coil oscillograph just discussed, is of immediate interest both for its own importance and as a vehicle for the study of important fundamental concepts of far-reaching significance.

The heart of the cathode-ray oscillograph is the ***cathode-ray tube.*** The tube comprises four principal elements as follows (see Fig. 3·3).

1. An ***electron gun*** ($KG_1G_2A_1A_2$) or a source for the rapid-fire propulsion of electrons through space in the form of a beam called a ***cathode ray.***
2. A ***control*** element (x, y) for redirecting the electron stream or ***ray.***
3. A fluorescent ***screen*** or target S to intercept and render visible the location of the ray.

4. A sealed ***container*** C for supporting the above parts as a rigid assembly in a modified atmosphere or vacuum.

The electron gun itself comprises four parts.

a. A cathode or electron emittor K of the heater type familiar in radio tubes.

b. A control electrode G_1 (grid 1) for controlling the beam current.

c. An annular accelerating electrode assembly G_2 (grid 2) and A_2 (anode 2).

d. A focusing electrode A_1 (anode 1) to concentrate the emitted and accelerated electrons into a beam or cathode ray of projectile electrons.

While the gun and screen merit considerable detailed study, it is the control (2) which, as before mentioned, is of immediate interest.

3·6. Control of Cathode Ray. Either or both of two types of control are possible. The cathode ray which comprises merely a stream of electrons can be deflected either *magnetically* or *electrostatically* according to basic source 3 or 1 respectively in our classification on page 35. These are represented by Figs. 3·3*a* and 3·3*b* respectively.

In Fig. 3·3*a* two small coils (xx) are supported just beyond the gun "muzzle," with their common axis perpendicularly intersecting the ray. A current I_x passed through the coils (in series) produces proportional density of magnetic field B_x and proportional deflection of the constant electron stream I by reason of the force $F = Bli$ acting on the portion l of the electron path through the field. The visible spot created on the screen is thus deflected in proportion to the magnitude and direction of the current I_x. By adding a second pair of coils (yy) with axis perpendicular to that of the first, the ray may be deflected also proportionally by a current I_y.

In this manner a graph of I_y versus I_x is produced in the familiar Cartesian coordinates. Such an instrument has innumerable possibilities when it is realized that either of these currents can be made to be a measure of any of many quantities electrical and otherwise. One current, say I_x, is commonly made to vary directly with *time* so that a picture of I_y versus time is obtained. $I_y = f(t)$ in these cases is generally cyclic or repetitive (a periodic function), and the timing current I_x must also be a periodic function of time in step or *in phase* with I_y. Because it is desired that I_x return to zero very promptly at the end of each time period or cycle, the graph of $I_x = f(t)$ is a saw-toothed curve: /|/|/|. The means for producing this current represent an interesting development in what are called ***"sweep circuits."*** Their study is best deferred until some knowledge of electronics is acquired.

Most cathode-ray oscillographs now employ the second or electrostatic control (Fig. 3·3*b*). Two parallel metal plates (xx) are supported just beyond the gun muzzle, closely astride of the ray, and consequently

within the tube. A potential difference E_x, applied to the plates so that one is positive and the other negative, will deflect electrons toward the former and redirect or "bend" the ray proportionally. A second pair of plates (yy) oriented about the ray 90 degrees from the first will permit a voltage E_y to deflect the ray and produce a graph of E_y versus E_x. The possibilities of the electrostatic or voltage control differ little from those already noted for the electromagnetic or current control.

3·7. The Electromagnetic Deflection of a Moving Electron. Turning now to the quantitative aspects of the deflection of the cathode ray, we shall first consider the case for magnetic deflection. The basic relation is that of the Oersted discovery, but it is concerned with the force on the individual electron rather than the current-carrying conductor as a whole. By substituting for dq in equation 3·1 the usual symbol e for the charge on the electron we have

$$F = B\,v\,e \qquad [3\cdot4]$$

This force is directed *transversely* or normal both to the direction of motion of the electron and to the direction of the magnetic field, as previously discussed. Because the force is normal to the direction of the field it cannot deflect the electron from a path normal to the field.

Furthermore the force, even after deflecting the electron from its original course, continues to be *normal* to the revised direction of the electron and never has a component in a direction to affect the speed v of the electron along its path. It follows that so long as the electron moves in a field of uniform density B, and with constant speed v normal to the field, the transverse force F will be constant (equation 3·4).

Since this force is always normal to the instantaneous direction of the electron, and is of constant magnitude when B and v are constant, the path of the electron in the field is an arc of a circle. This radial force is known in mechanics as a *centripetal* force and is readily expressed as follows.

$$F = \frac{mv^2}{R} \qquad [3\cdot5]$$

Equating equations 3·4 and 3·5 we obtain

$$R = \frac{mv}{Be} \qquad [3\cdot6]$$

Where R = radius of curvature of electron path (meters).
m = electron mass (9.11×10^{-31} kilogram).
e = electron charge (1.60×10^{-19} coulomb).
B = magnetic field density (webers per meter^2).
v = electron speed (meters per second).

The electron speed v is best found in terms of the accelerating potential V of the gun which is readily measurable. By equating the kinetic energy acquired by the electron to the potential energy used in the gun we obtain

$$\tfrac{1}{2}mv^2 = eV$$

$$v = \sqrt{\frac{2eV}{m}} \tag{3·7}$$

Substituting in equation 3·6, we find

$$R = \frac{m}{Be}\sqrt{\frac{2eV}{m}}$$

$$R = \frac{1}{B}\sqrt{\frac{2mV}{e}} \tag{3·8}$$

Substituting the values of e and m,

$$R = \frac{\sqrt{V}}{298{,}000B} \tag{3·9}$$

Let Fig. 3·4 represent the path of an electron projected from the gun G of a cathode-ray tube through magnetic field B to fluorescent screen

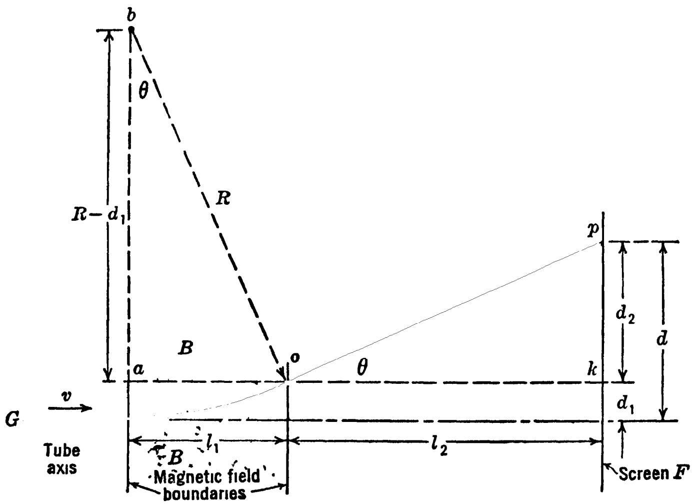

Fig. 3·4. Path of magnetically deflected cathode ray.

F. Let θ represent the angle subtended by the circular path of the electron pursued in the magnetic field. For the right triangle *oab* we write

$$R - d_1 = \sqrt{R^2 - l_1^2} \qquad [3 \cdot 10]$$

and

$$d_1 = R - \sqrt{R^2 - l_1^2} \qquad [3 \cdot 11]$$

Because *bo* is at right angles to *op*, the right triangle *pko* is similar to *oab* and the angle θ is contained by each as in Fig. 3·4. We may then write

$$\tan \theta = \frac{d_2}{l_2} = \frac{l_1}{R - d_1} \qquad [3 \cdot 12]$$

and

$$d_2 = \frac{l_1 l_2}{R - d_1} \qquad [3 \cdot 13]$$

Substituting equation 3·10 gives

$$d_2 = \frac{l_1 l_2}{\sqrt{R^2 - l_1^2}} \qquad [3 \cdot 14]$$

The total deflection at the screen then is

$$d = d_1 + d_2 = \frac{l_1 l_2}{\sqrt{R^2 - l_1^2}} + R - \sqrt{R^2 - l_1^2}$$

or

$$d = \frac{l_1 l_2 + l_1^2 - R^2 + R\sqrt{R^2 - l_1^2}}{\sqrt{R^2 - l_1^2}}$$

and

$$d = R + \frac{l_1 l_2 + l_1^2 - R^2}{\sqrt{R^2 - l_1^2}} \qquad [3 \cdot 15]$$

By substituting in this relation the value of R from equation 3·9, the value of $d = f(l_1, l_2, V, B)$ is readily found. The converse solution, for l_1, l_2, V, or B, is not readily performed with the exact relation of equation 3·15. For small angles of deflection where $R \geq 10l_1$, the l_1^2 term in the denominator of equation 3·15 may be neglected with little approxima-

tion, and the resulting relation clearly provides for ready solution of any desired factor.

$$d = R + \frac{l_1 l_2 + l_1^2 - R^2}{R}$$

$$= R - R + \frac{l_1 l_2 + l_1^2}{R}$$

$$d = \frac{l_1 l_2 + l_1^2}{R} \qquad [3 \cdot 16]^*$$

Substituting the value of R from equation 3·9,

$$d = 298{,}000 B V^{-1/2}(l_1 l_2 + l_1^2) \qquad [3 \cdot 17]$$

or

$$d = 298{,}000 B V^{-1/2} l_1 (l_2 + l_1) \qquad [3 \cdot 18]$$

It is to be remembered that this relation applies only when the angle of deflection θ is *small.* Many cathode-ray tubes are now so proportioned that θ is too large to permit the approximations involved in equation 3·18. If in doubt a check substitution of values in equation 3·15 is advisable.

3·8. The Electrostatic Deflection of a Moving Electron. The electrostatic control of a cathode ray depends only on the repulsion and attraction between electrostatic charges studied in physics.

Let Fig. 3·5 represent the path of the cathode-ray electrons from gun G to fluorescent screen F. Plates ab and $a'b'$ are made + and − respectively by the control potential E and produce an electrostatic field assumed uniform between the plates. The following symbols are used in addition to the lengths dimensioned in the figure.

m = electron mass (9.11×10^{-31} kg)
e = electron charge (1.60×10^{-19} coulomb)
V = accelerating (gun) potential (volts)
v = velocity of electron entering field (meters per second)
E = potential difference between control plates (volts)
s = spacing of control plates (meters)

* The approximate relation commonly given is $d = \dfrac{l_1 l_2 + \frac{1}{2} l_1^2}{R}$.

It is clear that the neglect of l_1^2 in the denominator $\sqrt{R^2 - l_1^2}$ of the exact equation 3·15 gives an approximate value of d for equation 3·16 which is too *small.* It follows that the use of $\frac{1}{2} l_1^2$ instead of l_1^2 would entail a still smaller and more approximate value than that derived here.

F_1 = transverse force on electron in field (newtons)
a_1 = transverse acceleration of electron in field (meters per second^2)
t_1 = time electron is in field (seconds)
t_2 = time of electron from field to screen (seconds)

The mechanical force exerted on a negatively charged particle in an electrostatic field is directed along the field toward the higher potential

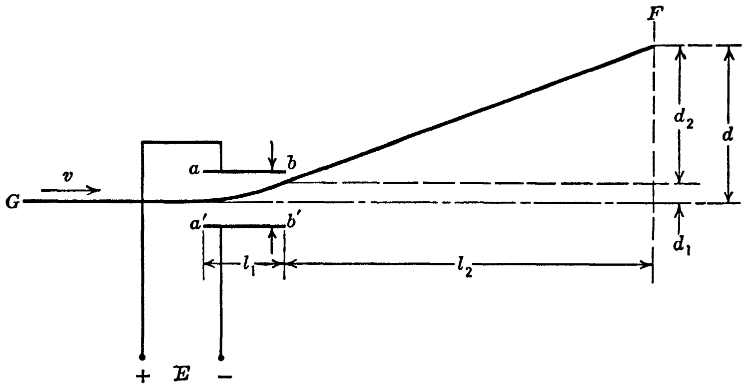

FIG. 3·5. Path of electrostatically deflected cathode ray.

or positive terminal of the field. It is directly proportional to the intensity of the field, as measured by its potential gradient, and to the amount of the charge. For an electron particle the relation is

$$F_1 = e\frac{E}{s} \qquad [3\cdot19]$$

The acceleration produced by this force is

$$a_1 = \frac{F_1}{m} = \frac{eE}{ms} \qquad [3\cdot20]$$

The consequent displacement or deflection effected by the passage of the electron through the entire field at a previously acquired axial velocity v is

$$d_1 = \tfrac{1}{2}a_1t_1^2 = \frac{eE}{2ms}\,t_1^2 = \frac{eE}{2ms}\left(\frac{l_1}{v}\right)^2 \qquad [3\cdot21]$$

The vertical or deflecting velocity imparted by the field is

$$v_2 = a_1t_1 = \frac{eE}{ms}\,t_1 = \frac{eE}{ms}\frac{l_1}{v} \qquad [3\cdot22]$$

As the electron leaves the space between the deflecting plates it has horizontal (axial) velocity v and vertical velocity v_2. The angle of deflection at which it leaves is then

$$\alpha = \tan^{-1} \frac{v_2}{v} \qquad [3\cdot23]$$

Assuming the electron to continue in a straight line to the screen, it will be further deflected by amount d_2 which is found by writing

$$\frac{d_2}{l_2} = \tan \alpha \qquad [3\cdot24]$$

Substituting equations 3·23 and 3·22 gives

$$d_2 = \frac{l_2}{v} v_2 = \frac{l_2}{v} \frac{eEl_1}{msv}$$

or

$$d_2 = \frac{eEl_1 l_2}{msv^2} \qquad [3\cdot25]$$

The total deflection, from equations 3·21 and 3·25, is

$$d = d_1 + d_2 = \frac{eEl_1}{msv^2}\left(\frac{l_1}{2} + l_2\right) \qquad [3\cdot26]$$

Substituting for v its equivalent from equation 3·7 in terms of V,

$$d = \frac{El_1}{2Vs}\left(\frac{l_1}{2} + l_2\right) \qquad [3\cdot27]$$

It is to be observed that the exact relations derived for deflection d in the foregoing depend upon two assumptions:

(*a*) The field is of uniform density.

(*b*) The boundaries of the field are sharp (no fringing) so that the electrons both enter and leave abruptly.

These assumptions also applied to the electromagnetic computation.

In contrast with the electromagnetic case it should be noted that the speed of the electron leaving the deflecting field is *not* the same as when it entered the field; it is $v_s = \sqrt{v^2 + v_2^2}$ instead of v. In further contrast the electrostatic field extends in some measure throughout the region traversed by the electron after leaving the space between deflecting plates and it is not to be assumed that the electron continues to the screen at velocity v_s. In fact there is no provision by which the deflecting plate potential can impart to the electron *permanently* the energy

$\Delta W = \frac{1}{2}m(v_s^2 - v^2)$ which the increased speed represents, and the conservation of energy principle must prevail.

It is well known that the electrostatically deflected beam does not remain in focus like the electromagnetically deflected beam for any considerable angle of deflection. This becomes so important for the comparatively wide-angle tubes used in television that magnetic deflection is the commercial practice for them although electrostatic deflection is the practice for ordinary oscillograph tubes. This defocusing for electrostatic deflection may be explained, even if the electron arrives at the screen only with its original velocity v, by presuming that it is not decelerated as rapidly as it was accelerated thus decreasing the time of transit from gun to screen.

3·9. Precautions with Cathode-Ray Oscillographs. Assuming that the auxiliary equipment used with a given cathode-ray tube is properly designed for it, this type of oscillograph is comparatively foolproof against damage by the operator. There are two precautions, however, which must be observed.

1. ***The beam must always be kept moving,*** otherwise the continuous bombardment of one spot on the fluorescent screen may "burn" a hole in it, i.e., destroy the fluorescence of the screen.

2. The tube is highly evacuated and ***to break it is dangerous,*** especially in the larger sizes. Just because the force is *inward* is no protection—the fragments acquire sufficient velocity to keep right on through center! Safety goggles are recommended when handling or working around exposed large tubes, and safety glass is sometimes installed in front of the screen.

The power pack voltage for the larger oscillographs commonly exceeds 1000 volts. Although this is not brought out to the panel of commercial oscillographs care must be exercised to avoid contact with it when exposed. *Even when switched off* certain capacitors in the pack may retain a really dangerous charge. ***Fatalities*** have occurred from this cause.

In taking photographs (oscillograms) from either type of oscillograph it is important to obtain auxilliary *calibration* traces of *known* voltages or currents for all values which are desired quantitatively. Because of film shrinkage and other difficulties, it is ***inaccurate to scale values*** off from the uncalibrated oscillogram.

3·10. The D'Arsonval Instrument. For most purposes the magnitude of a current or voltage, which may vary rapidly with time, need not be known as intimately as divulged by the oscillograph, and an instrument which indicates some kind of an *average* value suffices. One of the most common of such instruments, the D'Arsonval, employs the identical

principle of the Duddell or moving-coil oscillograph. The moving coil instead of comprising a single-turn loop mounted on bridges is of several turns wound on a rigid rectangular metal form which is free to rotate somewhat less than 180 degrees about a symmetrical axis as shown in Fig. 3·6.

It will be observed that two sides of the coil are free to traverse the confines of a permanent radial magnetic field of uniform density. These sides, conducting current i through and across the field of density B, receive the tangential force $F = Bli$ and are called the *active sides* of the

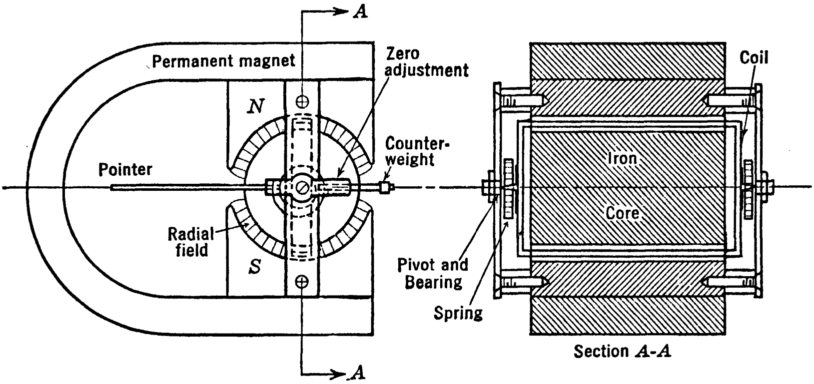

FIG. 3·6. A D'Arsonval instrument.

coil. The other or radial sides of the coil are termed *ends* and function merely as necessary mechanical and electrical accessories to the active sides.

If the radial air gap is uniform throughout the travel of the coil, the density B of the air gap flux will be uniform as in Fig. 3·6 and the torque $T = Fr = Blir = ki$, i.e., T will be directly proportional to the coil current. If, further, the spring torque is proportional to the angular displacement, the instrument scale will be uniform. This uniform scale usually is highly desirable. There are some exceptions, however. One of these is the familiar ***exposure meter*** which comprises a photovoltaic (light) cell connected to a D'Arsonval instrument. A standard D'Arsonval element would provide a uniform light scale. The photographer, however, is not so much interested in *arithmetic* difference of light as he is in *ratios*, and he would prefer a *logarithmic* scale. This is approximated by designing the air gap so that as the coil is rotated from zero it moves in a field of progressively less density (longer air gap). The torque then falls short of being proportional to the current or light, and the displacement for a given *increment* of current or light progressively decreases as desired.

Coil Supports. Two methods of supporting the coil are common. The more sensitive nonportable instruments known as ***galvanometers*** suspend the coil, with axis vertical, by a slender fiber or ribbon of gold alloy, copper, or phosphor-bronze. The ends of the supporting fiber are rigidly fixed to coil and frame of the instrument respectively so that angular deflection of the coil from normal position torsionally stresses the fiber and proportionally restrains the coil against the current-produced force. The principle involved of current-generated torque with spring restraint is identical with that studied for the Duddell oscillograph element.

Portable instruments must have a more rugged support for the coil. Two steel pivots are cemented, one to each end of the coil, and support the coil between jewelled bearings like those used with the balance wheel of a watch. The necessary spring restraint on the rotation of the coil is provided by two spiral springs, one at each bearing, again associated with the balance wheel of the watch except that nonmagnetic material must be used. Why *two* springs? The two springs, with torques opposed, provide a neat solution to the otherwise embarrassing problem of conducting current to and from a coil of this character. Furthermore, by balancing one spring against the other, temperature change should not throw the pointer off zero. The stationary end of at least one spring is attached to an adjustable member which provides a zero adjustment for the meter. This construction is a major feature of the original Weston patent.

Damping. It is elementary mechanics to observe that a coil restrained as described constitutes a torsional pendulum which may *oscillate* with a definite period or which may be damped to or beyond the critical or *nonoscillatory* status. In general considerable damping is desirable to facilitate rapid reading of the instrument and to steady the indication sufficiently to permit the reading of rapidly fluctuating currents. Either or both ***mechanical*** and ***electrical*** methods may be employed for this purpose. The mechanical comprises a vane or dynamic air brake moved by the coil in a partially closed chamber. The electrical exploits a tendency of the moving metal coil form to generate electrical energy and dissipate it in i^2r heat in the closed circuit comprised by the metal form. The motion of the coil is thus checked by dissipating its kinetic energy in heat electrically rather than mechanically.

Sensitivity. Instruments with built-in scales are commonly rated in ***ohms per volt.*** It must be clearly understood that this rating means ohms per volt of *full scale* and *not* ohms per *indicated* volt; the resistance, of course, is independent of the value of volts indicated by the instrument. It is recommended that ohms per volt be interpreted to signify

merely *reciprocal of full-scale current.* Questions involving the resistance of the instrument or of alterations in the resistance are analyzed from this point of view with less danger of confusion.

Until recent years ratings in excess of 1000 ohms per volt were unusual. Of late 20,000 ohms per volt ($\frac{1}{20}$ ma full scale) is readily achieved in rugged, portable, low-priced instruments. For many purposes the highest obtainable values of ohms per volt are quite unnecessary and it should not be inferred that the lower-resistance instruments are inevitably inferior or less desirable for all purposes. Note that ohms per volt provides no indication of the resistance of the coil itself although this may be useful and included among several data published by the manufacturer.

Magnets. The magnets are commonly of the horseshoe shape represented in Fig. 3·6. Tungsten steel has been used for many years, the essential property being permanence of magnetism even when subjected to considerable shock and to stray magnetic fields.

Individual instruments vary considerably in their susceptibility to stray fields. Even though the magnetic field of the instrument may not be permanently altered by a stray field it is important to guard against such fields and consequent error during the indication of the instrument.

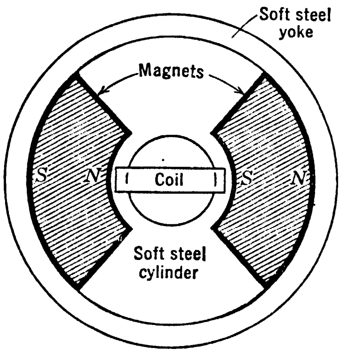

FIG. 3·7. Concentric magnet D'Arsonval instrument.

The design of D'Arsonval instrument magnets lately has been markedly altered by the availability of new steels with much higher retentivities. One of the most popular of these is Alnico, an alloy of aluminum, nickel, and cobalt, which permits retention of twice the flux density of tungsten steel, and with improved stability. The characteristics of these materials favor a short, stubby design rather than the comparatively long slender magnet of tungsten steel. This has led to the ***concentric-magnet*** design shown in Fig. 3·7. The symmetrical construction affords considerable shielding from stray fields which, as indicated above, is a feature of some importance.

Other alloys are now appearing which promise even higher retentivities at economically favorable costs.

3·11. Shunts for D'Arsonval Instruments. When it is desired to measure a current larger than can reasonably be conducted through the coil of a D'Arsonval instrument, a *shunt* is employed, either external or

built-in. The shunt provides an adequate path for the major part of the current, by-passing it around the D'Arsonval coil and sharing with the coil a strictly proportionate part in keeping with the rating of the instrument. The principle involved is merely that of Ohm's law as applied to two paralleled resistances.

It is desired frequently that one ammeter have several ***ranges,*** i.e., values of current for which the instrument may be connected to read full-scale deflection. Although it is possible and common to provide the desired shunt for each range with appropriate switching, as in Fig. 3·8, a more ingenious arrangement called the Ayrton shunt, which has practical advantages, is also widely used.

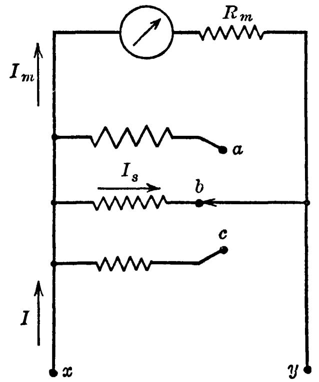

FIG. 3·8. Ammeter with conventional multirange shunt.

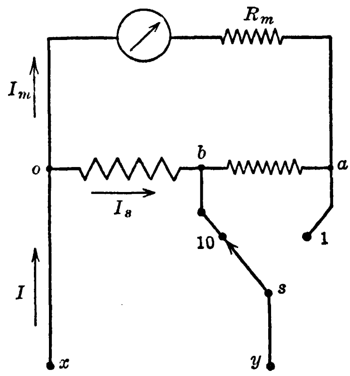

FIG. 3·9. Ammeter with Ayrton multirange shunt.

The Ayrton Shunt. The Ayrton shunt principle is represented in Fig. 3·9 which shows a D'Arsonval instrument of resistance R_m connected across a tapped shunt or series of shunts *abo* so that terminals *xy* may be connected either to *oa* or *ob*. With *xy* connected to *oa* the resistance R_{oa} is a conventional shunt with a ratio I/I_m determined as follows.

$$I_m R_m = I_s R_{oa}$$

$$I_s = I - I_m$$

$$I_m R_m = I R_{oa} - I_m R_{oa}$$

$$\frac{I}{I_m} = \frac{R_{oa} + R_m}{R_{oa}} \qquad [3\cdot 28]$$

With xy connected to ob the new ratio I'/I_m is found as follows

$$\begin{aligned} I_m(R_m + R_{ab}) &= I'_s R_{ob} \\ I'_s &= I' - I_m \\ I_m(R_m + R_{ab}) &= I'R_{ob} - I_m R_{ob} \\ I_m(R_m + R_{ab} + R_{ob}) &= I'R_{ob} \\ R_{ab} + R_{ob} &= R_{oa} \\ \frac{I'}{I_m} &= \frac{R_{oa} + R_m}{R_{ob}} \end{aligned} \qquad [3\cdot29]$$

If full-scale deflection is produced by current I_m in either of these cases we may observe that the ratio of the total currents required to give full-scale deflection for the two positions of switch s is

$$\frac{I'}{I} = \frac{(R_{oa} + R_m)/R_{ob}}{(R_{oa} + R_m)/R_{oa}} = \frac{R_{oa}}{R_{ob}} \qquad [3\cdot30]$$

By making $R_{ob} = 0.1R_{oa}$ we have $I' = 10I$ and the multipliers for the scale reading corresponding to the two positions of switch s are 1 and 10 as shown for positions a and b respectively. Additional taps to give multipliers of 100, 1000, etc., obviously are feasible and are not unusual.

It is to be noted in particular that the ratio I'/I does *not* depend on the meter resistance R_m. This is an important feature of this type of shunt and makes possible the marking or calibration of the shunt with the scale multipliers 1, 10, 100, etc., without regard for the resistance R_m of the particular meter with which it may be used. It is not to be inferred, however, that R_m is of no consequence; it does concern the relation between any of the currents I, I', etc., and the meter indication I_m, as shown in equations 3·28 and 3·29. Not infrequently it is desirable to pad the meter resistance by an external resistance in series with R_m of such value as will make the ratio I/I_m an integer multiple of 10 and simplify the interpretation of scale readings I_m into the desired values I.

Ayrton shunts like other instrument shunts of course are *rated* for a maximum permissible current. The several sections are economically graded or balanced in design so that the cross section of each is proportioned to the current it is expected to carry.

It should be noted by comparison of the circuits of Fig. 3·8 and 3·9 that the Ayrton shunt *avoids switch contacts* between the shunt and meter. This is an important practical advantage because the resistance

of switch contacts is sufficiently uncertain that their presence in the circuit of Fig. 3·8 may introduce error in the ratio I/I_m. Because the switch contacts of Fig. 3·9 are only in that part of the circuit outside of the shunt and meter combination they cannot affect the ratio I/I_m for the Ayrton shunt.

Especially when used with galvanometers the Ayrton shunt has the advantage of providing a practically constant resistance across the meter terminals *oa*. Because this resistance commonly controls the degree of damping of the moving coil, the use of the Ayrton shunt maintains a *constant damping factor* for the several shunt ratios.

Thus we find for the Ayrton shunt that

1. Calibration (ratio of ranges) is independent of meter resistance.
2. Ratio switch contacts cannot introduce errors in ratio.
3. Constant resistance across meter terminals maintains fixed damping for all ratios.

3·12. Multipliers for D'Arsonval Instruments. In general the basic D'Arsonval movement is applicable to measuring voltage as well as current. A ***voltmeter*** simply consists of the basic current-actuated instrument plus a built-in series resistor, a "multiplier," of sufficient current-carrying capacity and proper resistance to allow the flow of full-scale current when connected across the voltage desired for full scale. This is a simple Ohm's law relation.

$$I_m = \frac{V_m}{R_t}$$

For example, a 1-ma movement (1000 ohms per volt) with $R_m = 90$ ohms requires total resistance $R_t = 500/0.001 = 500{,}000$ ohms if it is to measure 500 volts full scale. The multiplier resistance must be

$$R_t - R_m = 500{,}000 - 90 = 499{,}910 \text{ ohms}$$

For the higher ranges, *external* multipliers are commonly supplied.

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QUESTIONS

3·1. Describe the physical situation which gives rise to the expression $F = Bli$.

3·2. Describe the *essential features* (parts and operation) of a cathode-ray oscillograph with electrostatic deflection.

3·3. Sketch an oscillogram of the sweep-circuit voltage and describe its function in an electrostatically controlled cathode-ray oscillograph.

3·4. Give two types of deflection control used for cathode-ray oscillographs and describe how they function.

3·5. Study the forces which act on the cathode-ray electron passing through the electrostatic and the magnetic deflecting fields and determine the shape of path pursued by the electron within each field. Identify each curve by *name*. (They are not the same.)

3·6. Explain for the cathode-ray oscillograph how it comes about that an approximate relation is desirable for computing the magnetic deflection of the electron beam but not for the electrostatic deflection.

3·7. Explain why the *magnetically* deflected cathode-ray oscillograph is usually preferred when the angle of deflection is relatively large.

3·8. What is the force (in pounds) of air against the screen of a 9-in. cathode-ray tube and what precautions are taken with it?

3·9. Describe the *essential features* (parts and operation) of a two-element Duddell oscillograph.

3·10. It is desired that the electromechanical converter or *vibrator* of the Duddell oscillograph give linear or exactly proportional response. What factors prevent the full achievement of this objective?

3·11. Discuss the relative merits and precautions required of each type (cathode ray and moving coil) of oscillograph for a given application.

3·12. Explain the significance of the designation *ohms per volt* and cite practical numerical values.

3·13. A 10-ma, 50-mv movement is to be used as a combination ammeter and voltmeter for two ranges of amperes and of volts. Draw a suitable circuit.

3·14. The *calibration* of the Ayrton shunt is not affected by changes in the instrument or meter resistance. Give rigorous proof.

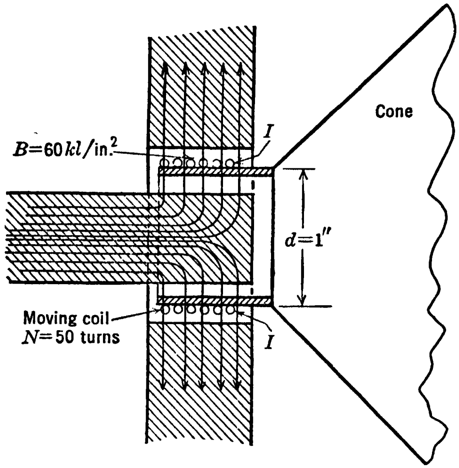
3·15. Make a sketch like the left-hand portion of Fig. 3·6 showing the modified structure used in the exposure meter including the *scale*.

3·16. A resistance measured with a voltmeter and ammeter is computed without allowing for the effect of instrument resistance. If the true value is higher than the computed value how were the instruments connected and which instrument reading was misleading?

PROBLEMS

3·1. Derive an expression for the radius R of the circular path of the electrons in the magnetic field of Fig. 3·4 as a function $R = f(m,v,B,e)$.

3·2. The moving coil of the dynamic speaker shown in the accompanying section operates in a field density $B = 60$ kilolines per square inch. At an instant when the coil carries 600 ma of current what electrodynamic force in ounces is produced?



3·3. Given a cathode ray tube with accelerating potential $V = 1000$ volts and dimensions as follows:

Width of control field along tube axis $l_1 = 6$ cm.

Distance from field to screen $l_2 = 37$ cm.

(*a*) If electrostatically controlled, what potential E must be impressed on control plates 1.2 cm apart in order to deflect the beam 7 cm on the screen?

(*b*) If magnetically controlled, what flux density B in webers per meter² is required to deflect the beam as above?

(*c*) If the above values of E and B are maintained, which control will produce the greater deflection when accelerating potential V is increased from 1000 to 2000 volts? By what per cent will the deflection be greater?

(*d*) What deflection on the screen can be obtained in (*a*) before the electron beam grazes the trailing edge of one electrostatic deflecting plate?

(*e*) In order to obtain the usual linear sweep action for the time axis, a sawtooth emf is impressed on one pair of the electrostatic control plates. If the upper frequency limit is determined by the requirement that any one electron must have sufficient time to traverse the entire deflecting field during one cycle or "tooth" of the sawtooth timing emf, compute this frequency.

3·4. A magnetically controlled cathode-ray oscillograph having dimensions $l_1 = 0.30$ in. and $l_2 = 20.0$ in. (Fig. 3·4) is to have a maximum beam deflection of 5 in. at the screen when the gun potential is 2000 volts.

(*a*) Compute by the approximate relation the field density B required.

(*b*) By the exact relation compute the deflection for the value of B found in (*a*) and determine the approximate per cent error in B introduced by the approximate relation.

3·5. The coil of a D'Arsonval movement has 60 turns of No. 36 copper wire and moves in a magnetic field of uniform density $B = 40$ kilolines per square inch which extends throughout 80 per cent of the length of either coil side. The area enclosed by each coil turn is a rectangle of 0.5 sq in. area.

(*a*) Derive a suitable formula for the coil torque in terms of the coil current and the given data.

(*b*) Compute the torque in inch-ounces for a current of 15 ma.

(*c*) What is the coil resistance at 20° C for a square coil?

(*d*) By what per cent will the coil resistance in (*c*) change for temperature −10° C?

3·6. A D'Arsonval type of d-c measuring instrument requires a current of 50 ma through the moving coil to produce full-scale deflection. The resistance of the coil is 2 ohms.

(*a*) What difference of potential across the coil will produce full-scale deflection?

(*b*) How can the above coil movement be used in a voltmeter to give scale deflection with 150 volts? Sketch the connection, and specify the required rating of the internal series resistance in *ohms* and *amperes.*

(*c*) How can the coil movement in (*a*) be used in an ammeter to give full-scale deflection with 5 amp line current? Sketch the connection and specify the required rating of the internal shunt in *ohms* and *amperes.*

3·7. A 25-amp meter has a resistance of 0.002 ohm.

(*a*) What is the voltage drop across the instrument for full-scale deflection?

(*b*) Specify the resistance and the current capacity of the external shunt to be used to read 100 amp line current with full-scale deflection.

(*c*) Show diagram of connections in (*b*).

3·8. A d-c voltmeter which has a resistance of 3000 ohms gives full-scale deflection with a potential difference of 150 volts across its terminals.

(*a*) How much current passes through the instrument?

(*b*) Show by diagram of connections how the range of the above voltmeter may be extended to give full-scale deflection when connection is made across 600 volts.

(*c*) What must be the resistance and the current capacity of the external multiplier to be used?

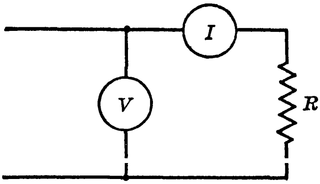
3·9. Two 0–150-volt voltmeters whose resistances are 15,000 and 5000 ohms, respectively, are connected in series across 200 volts. What should be the reading of each meter?

3·10. In order to determine the value of the insulation resistance between the field coils and the frame of a dynamo, a voltmeter is connected in series with this resistance across a 120-volt line.

A previous test showed a deflection of 30 volts when the meter was connected to the same supply in series with a known resistance of 45,000 ohms.

If the voltmeter reads 15 volts in the insulation test, what is the value of the unknown resistance?

3·11. A high resistance, measured as in the accompanying circuit, gives voltmeter reading $V = 120$ volts and ammeter reading $I = 0.300$ amp. The meter resistances are 10,000 ohms and 0.050 ohm respectively.



(*a*) Compute the per cent error which will result if the effect of the meter resistances is neglected. Is this negligible?

(*b*) What meter readings would be obtained in measuring this resistance from the same source of voltage if the voltmeter were connected directly across the resistance?

(*c*) What percentage of the true value would be obtained in (*b*) if the resistance were computed without regard for the meter resistance? Is the error negligible?

3·12. Resistances are to be measured by the drop of potential method with the following d-c instruments:

Voltmeter	150 volts, 1000 ohms per volt	
Ammeter	1.5 amp,	50 mv

The voltmeter is to be connected either directly across the unknown resistance or including the ammeter (as in Problem 3·11) according to which connection will give the least error for a particular resistance when it is computed simply as $R = E/I$, E and I being the respective meter readings without allowance for meter resistance effects.

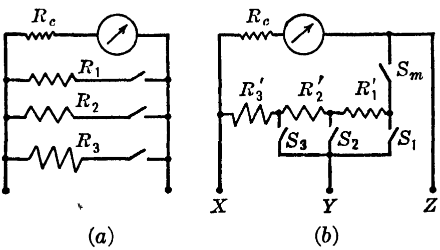
(*a*) What range of resistance can be measured by these instruments if readings only in the upper two-thirds of either instrument scale are of acceptable accuracy?

(*b*) Sketch the proper connections of meters for measuring low and high resistances respectively and explain.

(*c*) Compute the true value of the *transition* resistance, i.e., that resistance for which either connection in (*b*) will produce the same per cent error.

(*d*) Compute the per cent correction which should be applied to the measured value of the transition resistance. Is this the maximum per cent correction which will be required for the range in (*a*)? Explain.

3·13. A lecture room D'Arsonval ammeter was originally provided with three shunts (Fig. *a*) to give ranges of 1, 10, and 100 amp in addition to the unshunted 100-ma range.



R_c = coil resistance

R_1 = 1-amp shunt

R_2 = 10-amp shunt

R_3 = 100-amp shunt

It was found that this arrangement too readily invited injury to the meter by improper operation of the shunt switches. The shunts were reconnected as an Ayrton shunt; this connection minimizes the likelihood of injurious errors in switching. As in Fig. *b*, the 1-, 10-, or 100-amp ranges are obtained with S_m closed and

switch S_1, S_2, or S_3 respectively closed—terminals XY. To use the 100 ma-range, terminals XZ must be used with S_m open.

The original shunts R_1, R_2, and R_3 must be altered to values R_1', R_2', and R_3' for the new connection.

Compute the ratios R_1'/R_1, R_2'/R_2, and R_3'/R_3 and find the *per cent change* :o be made in each shunt resistance.

CHAPTER IV

MEASUREMENT OF RESISTANCE

4·1. Importance. The importance of resistance measurement to the electrical engineer need not be labored. From conductor values in microhms to insulation values in megohms he must know the resistance of innumerable kinds of apparatus. Among other purposes resistance measurements are made to disclose the *condition* of many kinds of communication and power equipment. By a schedule of periodic maintenance tests, progressive deterioration is often disclosed which, by timely repair, is prevented from causing not only expensive damage but also possibly serious interruption of service. As previously noted in Chapter II resistance is often measured to determine *temperature.* Numerous applications will appear in the course of later study.

Because an electrical resistor can be specified, constructed, maintained and measured with an exactitude and simplicity as yet excelled by no other electrical quantity it has been highly favored as a standard to go along with our standards of length, mass, and time. When the MKS system was internationally adopted in 1935 the electrical dimension was unfinished business. There is strong favor for the ohm, however, and the future may see adoption of the MKSO or meter-kilogram-second-ohm system. Standard resistors of high precision are commercially available and not uncommon.

4·2. The D'Arsonval Ohmmeter. The obvious procedure for measuring resistance is by use of a voltmeter and ammeter to obtain values of V and I in $R = V/I$ as exploited in the preceding chapter. If a voltage source is available of such constancy that it may be taken for granted without continued measurement, the voltmeter may be eliminated. The ammeter than reads inversely as the resistance and a new scale reading directly in ohms is possible. Ordinary dry cells provide emf's of sufficient constancy for the many commercial measurements where an error of a few per cent is unimportant. Instruments operating on this principle are now commonly available with self-contained battery and known as ohmmeters. They are made in a variety of size, range, and quality mostly for radio-receiver servicing and often with a switch or plug arrangement which permits the instrument to function at will as a voltmeter or milliammeter of several ranges according to the series or

shunt resistances which may be connected into the circuit in place of the battery. Various trade names such as multimeter, multitester, etc., are used with these combination voltage-current-resistance measuring instruments.

The circuit of Fig. 4·1 shows D'Arsonval movement of resistance R_c, battery E, and a resistor R comprising a simple ohmmeter with terminals xx for measuring a resistance such as R_x. Resistance R is required to limit the current within the range of the D'Arsonval element even when terminals xx are connected so that $R_x = 0$. If I_{max} is the maximum scale value this resistance evidently must be $R = (E/I_{max}) - R_c$.

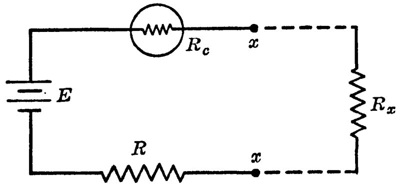


FIG. 4·1. A D'Arsonval ohmmeter circuit.

The Scale. The scale point for I_{max} thus becomes the zero point for the resistance scale. The instrument current in general is expressed by $I = E/(R + R_c + R_x)$. To obtain $I = 0$ will require R_x = infinity, and the scale point for zero current becomes infinity for the resistance scale. The range of the ohmmeter apparently is zero to infinity. The R scale, unlike the I scale, is not uniform or evenly divided, however.

Expressing the constant factors by a and b, the equation for $R_x = f(I)$ becomes $R_x = (a/I) - b$. This is clearly a *hyperbolic* function. R_x will increase more rapidly for the smaller than for the larger values of I and, for an instrument with the usual uniform scale of I, the resistance scale will progressively crowd toward the infinity end of the scale. Consequently there is for each instrument a practical upper limit of readability.

Some sense of the practical range of an ohmmeter is readily obtained by computing the value of resistance R_x which produces mid-scale indication or current $\frac{1}{2}I_{max}$. Since $I_{max} = \frac{E}{(R + R_c)}$ it follows that $\frac{1}{2}I_{max}$ $\frac{E}{2(R + R_c)} = \frac{E}{R + R_c + R_x}$ and $R_x = R + R_c$. In other words ***the mid-scale value of resistance must be equal to the total internal resistance*** of the ohmmeter. This is a cardinal requirement for these instruments but it is not always observed in the manufacture of the cheaper varieties.

Compensation. Long before the battery voltage falls to a degree that warrants discard, the voltage consistently and slowly decreases so that the indication for $R_x = 0$ will depart from scale zero more than can be tolerated. It is necessary in the design of a practical instrument to

r...

nected as sh...

The bridge is usu...
by what is called the ***null me...***
i.e., it is *balanced* so that no potential exists across ab. With a sufficiently sensitive galvanometer the balance can be determined with high precision. ... ance is readily obtained from Ohm's law together that $I_1 = I_2$ and $I_x = I_3$ when $I_{ab} = 0$ (balanced).

FIG

Since $E_{ab} = 0$,

$$I_x R_x = I_1 R_1$$

and

$$I_3 R_3 = I_2 R_2$$

Dividing one equation by the other and cancelling equal currents,

$$\frac{R_x}{R_3} = \frac{R_1}{R_2} \tag{4·7}$$

and

$$R_x = R_3 \frac{R_1}{R_2} \tag{4·8}$$

The value of R_x is found in terms not of emf or current but of the other resistors R_1, R_2, R_3. One resistor, such as R_3, must be of the absolute precision to which R_x is required. The other two appear only as a ratio R_1/R_2 and their absolute values are unimportant. It is relatively easy to produce resistors with high precision *ratio* as compared with the task of assuring high *absolute* precision. These resistors R_1/R_2 comprise what are called the ***ratio arms*** of the bridge.

The bridge offers no exception to the general rule that *thermal or contact emf's must be guarded against* in precision measurements. Pro-

e.
par-

perated for null
ction line, for example,
value of the resistors and the
zed to measure the deviation of a
y correlating galvanometer deflection
erance limits can be marked on the gal-
control of the resistors established within the
mputation of the unbalanced bridge will be con-
hapter on networks.

Method. When conductors or resistors carry alter-
the density of the current is likely not to have the same
ver a cross section of its flow as it would have for unvary-
ct current. The effective resistance of the path is commonly
r for alternating current than for direct current and requires the use of a more complicated a-c bridge or of other methods for its measurement. In these cases it is sometimes advantageous to utilize the relation $P = I^2R$ and to determine R from the heating power P and the current I as measured by a-c wattmeter and ammeter respectively. The relation $R = P/I^2$ is now given in the AIEE Standards as basically defining electrical resistance. In the simpler cases $R = E/I$ is equally good but must be used with care to insure that the voltage measured is really and exclusively that associated with the ohmic drop IR.

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QUESTIONS

4·1. If a standard milliammeter is used for a D'Arsonval type ohmmeter, show where on the scale of milliamperes the *zero* and *infinity* of the new ohm scale should be.

4·2. Explain what factors determine the mid-scale reading of the ohmmeter in question 1.

4·3. Describe three ways by which the D'Arsonval type ohmmeter is provided with an adjustment to compensate for battery aging and explain why each is only *approximate.*

4·4. If the battery in Fig. 4·2 acquires appreciable resistance as it ages to the minimum voltage for which compensation is provided, determine whether the change in circuit resistance produced by adjusting the compensator is in such direction as to be beneficial or otherwise to the accuracy of the ohmmeter.

4·5. Derive the mathematical function $R = f(S)$ representing the values of resistance R as a function of distance S along the scale of the D'Arsonval type ohmmeter and determine from analytical geometry whether it represents a parabolic, hyperbolic, etc., distribution of R values.

4·6. Explain clearly why the megger ohmmeter in contrast with the D'Arsonval ohmmeter is comparatively independent of the emf used with it.

4·7. Explain why, in contrast with the usual D'Arsonval movement, restraining springs for the moving coils are unnecessary in the megger ohmmeter.

4·8. Explain why a bridge may be preferred to the ohmmeter type instrument for resistance measurement.

4·9. Explain why and how bridges are operated *un*balanced.

4·10. For what reason is resistance defined by the relation $R = P/I^2$ in preference to $R = E/I$?

PROBLEMS

4·1. What percentage of maximum rated current should flow through a 20,000 ohm per volt ohmmeter with the circuit of Fig. 4·1 when $E = 45$ volts and $R_x = 30$ megohms?

4·2. (*a*) Compute the value at which the 0–400-ohm compensating rheostat should be set in Fig. 4·2 when the battery voltage is actually 4.50 volts as labeled and adjustment for $R_x = 0$ is being made. Check the total ohmmeter resistance and determine whether the 3470-ohm resistor is correctly chosen.

(*b*) Compute the extreme values of battery voltage for which the rheostat is designed to compensate.

(*c*) What per cent correction should be applied to the 3500-ohm reading for each extreme value of battery voltage in (*b*)?

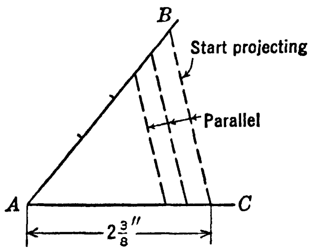
4·3. Given that the ohmmeter in Fig. 4·2 could be read satisfactorily if scale divisions were spaced as close as $1/150$ of the total scale length,

(*a*) What, conservatively, are the highest and the lowest value of resistance which should be *marked* on the scale between zero and infinity?

(*b*) For what resistance range would it be fair to rate the instrument?

4·4. The compensation provided for different battery voltages in the circuit of Fig. 4·2 provides no adjustment for maintaining the total internal resistance of the ohmmeter to the 3500-ohm value without which it cannot read correctly. If 70 ohms of the 3470-ohm resistor is made adjustable by means of a 0–70-ohm rheostat what combination of maximum battery resistance and minimum battery voltage can be fully compensated for by this *dual* rheostat adjustment? Assume that the battery resistance is independent of the current for any values required by the ohmmeter.

4·5. It is desired to construct a D'Arsonval ohmmeter, to read 4500 ohms at mid-scale, using a 1-ma (full scale), 105-ohm D'Arsonval meter (Weston 301 type). The milliammeter scale is divided into fifty equal parts. The length of the scale arc is $2\frac{3}{8}$ in. A two-terminal, 4.5-volt battery is to be used.



(*a*) Draw the circuit diagram of the ohmmeter and specify the resistance R to be used in series with the milliammeter.

(*b*) Lay off the meter scale as a straight line and locate accurately by computation the points to be labeled 500, 1000, 2000, 3000, 5000, 10,000, 20,000, 50,000, 100,000 ohms. It is well to lay off the computed distances to any convenient scale (English or metric) which may have a total length (AB) between 3 in. and 5 in. and then project these onto the correct length AC.

(*c*) If series rheostat compensation for decrease in battery voltage up to 5 per cent is to be provided, what range of adjustment of R is necessary?

(*d*) When the maximum adjustment described in (*c*) is made for low battery voltage, what per cent correction factor should be applied to the 4500-ohm division at the midpoint of the scale to compute the true resistance?

(*e*) A scale is to be added having a midpoint reading of 45 ohms. Sketch the revised circuit and show all resistance values. Retain the original series resistance R with value found in (*a*). Remember to check the scale point corresponding to $R_x = 0$. Study the operation of the meter with the new scale and discuss the practicability of the arrangement.

4·6. An inexpensive Wheatstone bridge is provided with ratio arms each adjustable only to the values 1, 100, and 1000 ohms, and each good to ±0.1 per cent. The third arm is adjustable, in 1-ohm steps, from 1 to 10,000 ohms, each good to $\pm\frac{1}{2}$ per cent.

(*a*) Compute the resistance range of the bridge.

(*b*) Assuming perfect balancing of the bridge, how many significant figures (the last one doubtful) are permissible in the result?

(*c*) When the ratio arms are set for 100 and 1000 with the latter opposite the unknown, and the third arm reads 6427, compute the unknown showing the proper number of significant figures.

CHAPTER V

ELECTROMOTIVE FORCE

5·1. The Nature of Electric Potential. When a quantity of water dq is displaced so that it experiences a rise or fall in pressure or *head* of amount h, a change $dW = h\,dq$ occurs in its potential energy. This relation is familiar as used in computing the energy or power available from a waterfall and in many other problems.

Let us consider the hydraulic circuit of Fig. 5·1. The energy acquired by a unit of water dq as it traverses the pump from b to a is $dW = h\,dq$ where h is the increase in head or potential imparted to dq by the pump.

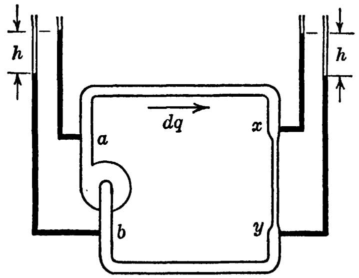


FIG. 5·1.

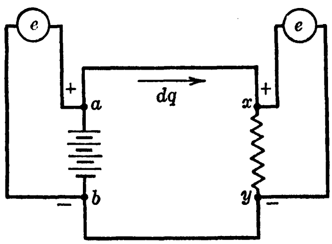


FIG. 5·2.

Showing analogous hydraulic and electric potentials.

Let us consider the entire friction head in the pipe to be included in the drop h at the constricted section xy which we shall call the *receiver.* The same relation $dW = h\,dq$ with negative h gives $-dW = -h\,dq$ and indicates that the same energy acquired by dq at the pump is delivered to the receiver during the traverse of dq from x to y. Clearly the water functions here as an agent or medium for transferring energy from pump to receiver. We might visualize an endless caravan of trucks dq each loading dW at ba and dumping at xy as it traverses the loop.

5·2. Energy Concept of Potential. From this process we may gain some concept of the nature of this entity h which we call head or potential. By transposing our energy equation to give $h = dW/dq$ we find that h is simply a measure of the amount of energy dW that the pump

can pack into a unit quantity of water dq. In terms of the truck caravan it might be called the *loading factor*, i.e., the amount of "energy" piled onto each truck as it traverses the loading platform *ba*.

This concept of the significance of hydraulic head or pressure applies to potentials in general. Turning to Fig. 5·2 we find much the same story for the analogous electric circuit. The electrons which traverse the circuit convey energy from battery to resistor. They are loaded with energy $dW = e\,dq$ as they traverse the battery through potential e and are relieved of their burden as they experience fall in potential while traversing the resistor.

Thus we observe that the voltage, emf, or electric potential difference e between the terminals of any circuit element indicates the *amount of energy dW which it can give to or take from a unit carrier dq.*

When a battery has an emf of 3 volts, for example, it signifies that the battery has the ability to deliver electrical energy to electron carriers at the rate of 3 joules per coulomb of carrier. In other words 3 volts indicates that the battery is potentially capable of turning out electric energy "packaged" 3 joules per coulomb; it may or may not be in the act of doing so.

When it is clear that the emf of an apparatus indicates its potency for energizing or de-energizing electrons, the significance of the now commonly used ***electron volt*** as a unit of energy is apparent. Since 1 coulomb-volt = 1 joule and 1 coulomb = 6.3×10^{18} electron charges, 1 joule = 6.3×10^{18} electron volts.

5·3. A Polarity Relation. It should be observed that the energy aspect of electric potential is often useful in determining its polarity. The electrons displaced around an electric circuit must experience as much rise as fall in potential during a round trip; otherwise the conservation of energy principle is violated. *When it is known that a circuit element ab is an* ***energy source*** *(a loading station for the energy carriers) the conventional current flow must be from* $-$ *to* $+$ *through this source.* The converse of course is true for a ***load*** element. Attention is directed to this feature because too often the student is misled by an erroneous belief that current is always directed from + to − or *down*hill. Electricity like water can and must flow *up*hill from − to + in some part of every circuit, otherwise how does it get *up* to come *down* during the round trip?

5·4. Emf versus Potential Energy. The foregoing significance of emf must not be confused with either the total energy resources of a source of emf or with the time rate at which it may deliver energy. The emf of static electricity is of the order of kilovolts yet stroking the cat produces little electrical energy and is rarely fatal to either partici-

pant from direct electrical causes. Conversely the low emf of a well-charged 6-volt automobile storage battery gives no indication that the battery contains enough energy to send a 200-lb man about 1½ miles into the air or to make fatal progress in roasting him. Neither does it indicate how long either alternative might require. Let it be clear that *potential is **not** synonymous with potential energy.* We shall observe presently that some sources of emf are incapable of conducting any displacement of electricity at a rate sufficient to deserve being called a current.

5·5. The Potentiometer. Although the D'Arsonval and other types of indicating instruments are suitable for the measurement of voltage

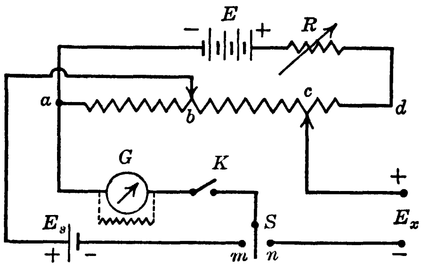


FIG. 5·3. A simple potentiometer circuit.

with the precision *usually* required, it is not uncommon to require precision of a higher order comparable with that obtained in the measurement of resistance by the bridge method. For this purpose we use an instrument called the potentiometer.

The basic circuit is shown in Fig. 5·3. The potentiometer operates by what is known as the ***Poggendorf method.*** This consists in comparing the emf E_s of a precision standard cell with an unknown emf E_x by balancing each against the appropriate length of a carefully calibrated resistor *ad* which is carrying steady current provided by a third source of emf E as a power supply.

There are four principal elements as follows.

1. The standard cell E_s.
2. A calibrated resistor *ad* with adjustable contacts *b* and *c*.
3. A power supply E with control R for producing a sustained voltage drop across the resistor *ad*.
4. A sensitive galvanometer or "detector" with mid-scale zero for determining the desired null adjustment.

With switch S on m, momentary comparison of E_s with V_{ab} is had by momentarily closing key K. As advised by the deflection of galvanom-

eter G, adjustment of R is made until null indication of the galvanometer shows that $V_{ab} = E_s$. Switch S is now changed to n and contact c is adjusted until $V_{ac} = E_x$ as denoted by null indication of the galvanometer.

Then

$$\frac{E_x}{E_s} = \frac{V_{ac}}{V_{ab}} = \frac{IR_{ac}}{IR_{ab}} = \frac{R_{ac}}{R_{ab}}$$

or

$$E_x = E_s \frac{R_{ac}}{R_{ab}}$$

Since ad is a resistor calibrated for all values of R_{ab} and R_{ac}, and since E_s is known, E_x is determined.

It is usual to design the potentiometer so that the scale of values traversed by contact c will indicate directly in volts for E_x and computation is thus avoided. Typical ranges of the commercial instrument are 0 to 1.61, 0 to 0.161 and 0 to 0.016 volt. Higher ranges are obtained by external use of another calibrated resistor commonly called a *volt box*. The galvanometer is usually provided with one or more shunt (or series) resistors to reduce sensitivity during the preliminary stages of balancing. The resistors function of course in the manner already studied for D'Arsonval instruments.

It will be observed that both the standard cell and the unknown emf are *not* required to carry any current when the final adjustment which measures each emf is accomplished and that no considerable current need flow at any time if the instrument is properly operated. The cell commonly used is the Weston standard cell which is constructed with great care to insure permanence of emf when properly used. The emf of these cells is of the order of 1.018 to 1.019 volts, usually around 1.0183 volts. Contact b for the standard cell is usually a series of taps accurately calibrated and labeled for connection in accord with the particular value marked on a given cell. With b on the proper tap, adjustment of R for null reading of G insures that E is supplying through $abcd$ the current for which the scale of contact c is calibrated. These contacts may comprise taps or a slide wire or both where the latter is the fine adjustment.

The price of commercial potentiometers ranges from about \$50 to \$1500.

5·6. Sources of Emf. Because emf's are intimately associated with potential sources of electric energy, the principal sources of emf are

logically classified and considered according to the *kinds of energy* involved, as follows.

1. *Mechanical.*
   a. Induction by motion—electromagnetic and electrostatic.
   b. Piezoelectric effect—distortion of certain crystals, notably of quartz and Rochelle salt.
   c. Volta effect (contact emf)—friction by rubbing, adhesion, or cleavage.

Of (*a*) our knowledge is quite satisfactory for engineering purposes. Practically all electrical energy for power purposes is obtained from mechanical energy by electromagnetic induction. The laws of both electromagnetic and electrostatic induction are well known and, in the next chapter, will be considered in some detail.

Of (*b*) our knowledge is less exact. We do know enough about materials which possess this property of producing a small emf when distorted by mechanical pressure so that numerous practical applications have been developed since the advent of electron tube amplifiers with considerable "gain" or voltage-amplifying ability. The piezo crystal is inherently a voltage-producing device and is useful only in translating small mechanical displacement or pressure into an electrical emf replica. The present crystal microphones and crystal phonograph pickups are typical applications of the *Rochelle salt* crystal. From these, emf's of the order of 10 mv and up, and powers of the order of 10^{-10} watt are obtained. Permanent injury is invited unless care is used to seal the crystal against *moisture*, and to avoid *temperatures* in excess of approximately 100° F (not C). Other crystals such as *quartz* and *tourmaline* are more rugged but their output is but a fraction of that of Rochelle salt. They have been used with some success in the study of internal-combustion engine pressures, vibration studies, etc. Quartz crystals are widely used in the precise control of frequency for radio transmission and for time standards.

Of (*c*) a modest amount of quantitative data has long been available. It is known that a small emf is produced between the faces of any two conductors of different material which are in contact. The magnitude of the emf depends upon the materials, temperature, and the intimacy of contact but apparently not upon the area or shape of the contact. The relation appears to be linear in that the contact potential between any two of three materials is the algebraic difference of their respective contact potentials with reference to the third material. For example, a copper-iron contact gives $E = (0.71 - 0.56) = 0.15$ volt with polarity of Cu negative.

The accompanying table of potentials is given with reference to zinc. Zinc is positive with respect to all the other metals here listed.

Zinc with Respect to	Emf—Volts
Antimony	0.41
Bismuth	0.49
Copper	0.71
Iron	0.56
Lead	0.15
Nickel	0.47
Platinum	0.88
Silver	0.91
Tin	0.25
Carbon	1.10

The principal application of knowledge of contact potentials is in a negative way, i.e., to avoid errors which such potentials may introduce in the use and measurement of small emf's.

(The above contact potentials between dry metals are not to be confused with contact potentials of electrodes in electrolytes, which were also discovered by Volta.)

2. *Thermal.*

   a. Seebeck effect—unbalance of the normal emf's of contact (volta effect) at junctions of dissimilar metals by holding the junctions at dissimilar temperatures.

   b. Thompson effect—production of electric potential difference along a current-carrying conductor which is not at uniform temperature along the current path, i.e., temperature gradients along the current path are accompanied by electric potential gradients.

   c. Pyroelectric effect—heating of certain crystals, notably of quartz and tourmaline.

Although none of these are sources of appreciable power, the thermoelectric effect (*a*) discovered by Seebeck in 1822 is of outstanding practical importance. It is widely exploited in what are called ***thermocouples*** for the measurement of temperature. One junction of the pair of dissimilar wires is maintained at constant known temperature (commonly in ice water) and the other junction is located at the unknown temperature. The emf, preferably measured by a potentiometer, is a measure of the unknown temperature. The thermocouple is especially useful where the temperature to be measured is inaccessible for thermometers, as within the coils of a transformer or motor, or where the temperature, temperature range, or rate of change is excessive for ordinary thermom-

etry, or where high precision is required. Temperature tests on electrical equipment sometimes utilize hundreds of thermocouples embedded in strategic locations.

The extensive practical use of the Seebeck effect for precise measurement of temperature has provided fairly reliable data covering the more commonly used materials. These data are usually presented in ***calibration tables*** of the emf versus temperature of the "hot" junction of a given pair of metals or *elements* when the "cold" junction is maintained at 0° C.

For precise work it is best to check the calibration of the particular couple by *actual measurement* of the emf at several known temperatures throughout the range in which it is to be used. This is done by immersing the hot junction in various liquids for which the freezing and (or) boiling points are well established, the cold junction temperature being maintained as intended during actual use.

The choice of thermocouple materials is governed by numerous considerations but especially important among these are, of course, the *temperature range* and *chemical composition* of the material to be measured.

Typical data for four of the most commonly used thermocouples are plotted on the graph of Fig. 5·4.

These couples may be used bare for many applications but protecting tubes of a suitable metal or ceramic are commonly used. The Chromel-Alumel and platinum couples, for example, must be well protected from chemically *reducing* atmospheres. Wire sizes from No. 30 to No. 8 AWG are common, the smaller sizes being permissible for the lower operating temperatures.

Vacuum Thermocouples. If a small resistor is mounted in contact with a thermocouple so that heat may be developed by passing current through the former, the latter will generate an emf which may be measured with a suitable D'Arsonval instrument. The instrument indication is a *measure of the current* in the resistor and when so calibrated the combination provides a sensitive ***thermoammeter.*** These instruments are particularly suitable for the measure of high-frequency currents and voltages, values up to 10 kilocycles being readily handled, and, with proper care, even up to 2000 kilocycles. They are also useful where the plain coil-type instruments are either insufficiently sensitive or where the power taken by such instruments would unduly upset a low-power circuit. The overload capacity of these thermocouple-resistor units is *very little* so that exceptional care is necessary to avoid burn-out.

The vacuum mount is for the purpose of reducing the heat loss so that the power required may be kept to a practical minimum. This

eliminates convection, but conduction and radiation losses continue. The heating of course depends on the square of the current (I^2R); the conduction loss is directly proportional to the temperature, and the radiation loss to the fourth power of the temperature.

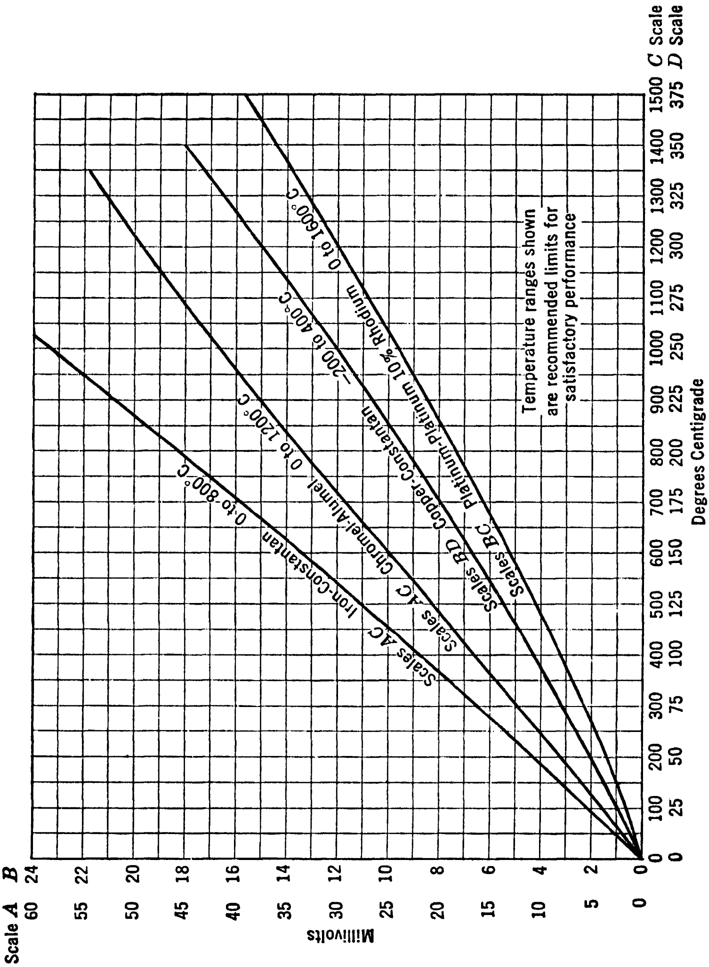


Fig. 5·4. Typical thermocouple characteristics.

The temperature (and thermocouple emf, Fig. 5·5) therefore, starts to increase very nearly parabolically, according to the square of the current (OA), but eventually the more rapidly increasing radiation loss predominates and slows up the rate of temperature rise even to the extent finally of reversing the curvature (BC). By suitable design it has been found possible to produce a reasonably *linear* relation between thermocouple emf E and measured current I over most of the operating range

such as AB (Fig. 5·5). This provides the advantage of a nearly uniform scale over most of the instrument range. The maximum operating temperature of the thermocouple at B approximates 300° C and the emf about 15 mv.

To get some idea of size it may be observed that the glass bulb for these thermoelements ranges from a tube of about ½ in. diameter by ¾ in. long to a 1-in. diameter sphere. These are commonly made to plug in for ready replacement either in the event of damage or for substitution of a different ampere range. Ratings run from about 1.5 ma 800 ohms, to 500 ma 0.3 ohm, and prices from $25 to $35.

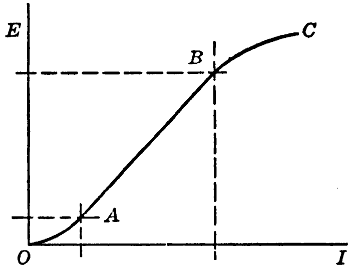


FIG. 5·5. A vacuum thermocouple characteristic with linear range AB.

Nonvacuum types are available up to 50 amp rating and at lower prices. The construction of these, of course, bears little resemblance to the vacuum type just described.

3. *Chemical.* Volta effect—contact potential by immersion of dissimilar electrodes in liquid electrolyte.

The principles involved in the electrolytic Volta effect and their application are beyond the scope of this study.

4. *Electrostatic.* Electrostatic induction through electrostatic coupling.

The laws of electrostatic induction by coupling are well known and will later be considered. The principle is usefully applied mainly in electron tube equipment. It is of broader importance in a *negative* sense; in both power and communication equipment it is likely to appear where not wanted, and the means for its elimination sometimes present a difficult practical or economic problem.

5. *Magnetic.*

  a. Electromagnetic induction through magnetic coupling.
  b. Hall effect. A rectangular metal plate in and incident to a magnetic field has a voltage between two parallel edges when current flows between the other two edges.

Item (*a*) is the well-known principle employed in transformers, induction coils, and the like. It will later be considered in detail.

The Hall effect (*b*) produces emf's which are very small and which so far have had little practical significance in engineering work.

6. *Electromagnetic radiation.* Photovoltaic effect—generation of emf between two surfaces incident to electromagnetic radiation (light).

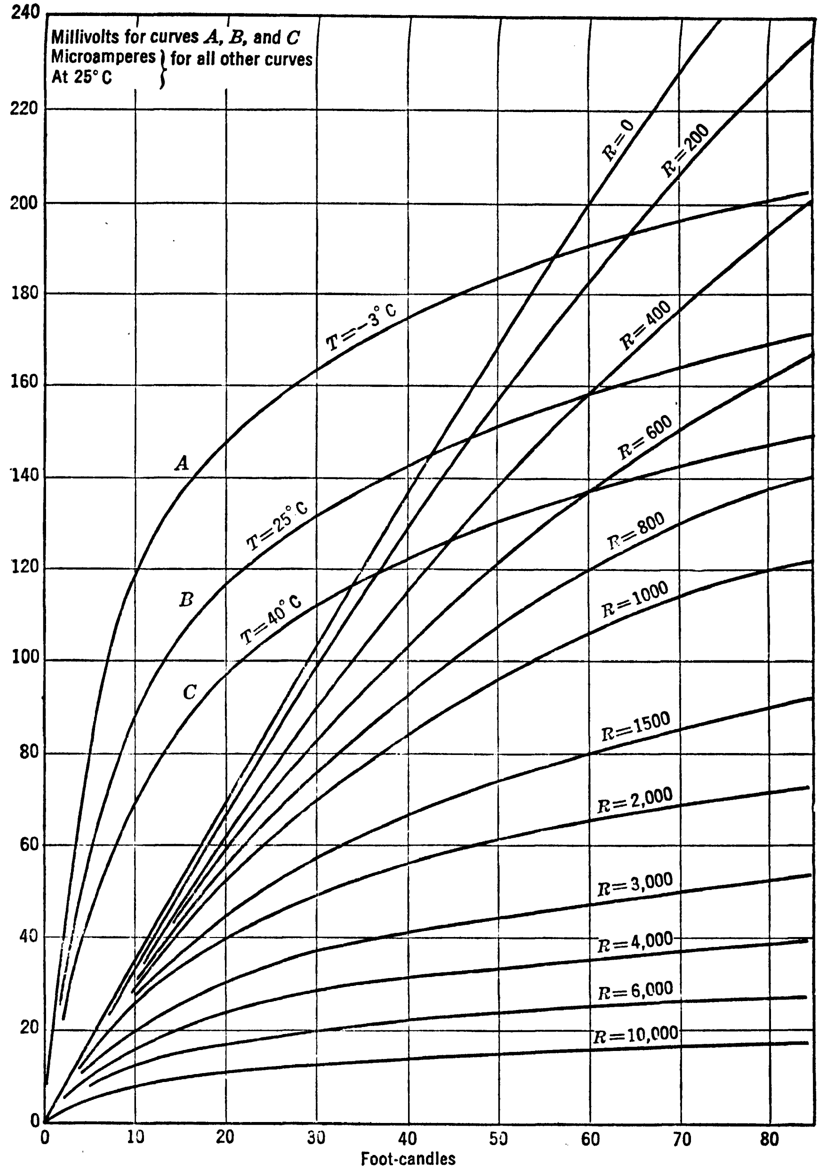
It has long been known that light, falling upon the surface of contact between certain combinations of liquid and solid conductors, would generate a small emf more or less in proportion to the amount of light reaching the contact surface but independent of the area of the contact surface. In general these same combinations have the useful property of *rectification*, i.e., they conduct better in one direction than the other across their contact boundary.

In 1926 Grondahl was working in the Westinghouse laboratories with a non-liquid or "dry" variety of this contact-type rectifier, comprising a surface of *cuprous oxide on copper*, when he discovered that the rectifying action was affected by light. This led to development of the convenient dry-type photovoltaic cells which are now extensively used in the measurement of light and in the operation of electronic control equipments which depend upon light as the primary element. Other materials than Grondahl's were found to produce the photovoltaic emf. The *selenium-iron* cell has dominated the market in this country. Improvements in performance have been achieved but, in the absence of adequate fundamental knowledge, they have mostly come by the "hard" way of lucky experimental accidents.

The curves of Fig. 5·6 are given by the General Electric Co. for their ***selenium-iron cell*** which is quite similar to the Weston cell. These cells consist of a steel plate 1.7 in. by 0.9 in. by 0.065 in. on which is deposited a layer of selenium and, for electrical contact with the selenium, two transparent layers of conducting metal which provide the negative electrode. The whole is covered with a protective coating and mounted in a Textolite case with a glass window.

Photovoltaic cells are beset by numerous influences which jeopardize their use for accurate measurements. The emf is not only a function of the *amount* and *wavelength* of incident light but also of *temperature, resistance* of circuit to which connected, *time-rate of change* of light, *angle of incidence* (other than normal cosine function), and *humidity.* Individual cells of the same manufacture vary somewhat in their performance characteristics and all general data given by manufacturers is only average or typical.

It will be noted that the current produced by the cell on *short* circuit is *directly proportional to the incident light.* On *open* circuit the emf is a *logarithmic* function of the light. It is usual to operate the cell with a load of sufficiently low resistance to approximate linear response and to minimize the temperature effect which is possibly negligible at some low value of resistance but which is by no means negligible on open circuit. The useful life of these cells is indefinite and seems not to be a function of the amount of use. Moisture, mercury vapor, and excessive tem-



Courtesy of General Electric Co.

FIG. 5·6. Typical curves for a General Electric light cell. Open circuit millivolts at different temperatures *A*, *B*, *C*. Microamperes at 25° C for different resistances in series with cell.

perature (over 50° C) are their outstanding enemies. It is to be clearly observed that *no external source of emf* is required for the use of these cells and is to be strictly avoided except as definitely permitted by the manufacturer for very special circumstances.

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QUESTIONS

5·1 Explain concisely in your own words your concept of the nature of electric potential. How related to electric energy? Analogous to what quantities in mechanics, hydraulics, and thermodynamics?

5·2. High voltage is generally considered dangerous. Under what circuit conditions is it relätively safe?

5·3. There is some doubt whether the pyroelectric emf is actually different than a piezoelectric emf. Explain how this can be.

5·4. Given a potentiometer made of inferior material, so that the calibrated resistors are not reasonably immune to temperature changes, let it be in a hot room where *all* the instrument resistors increase 1 per cent from normal value before use of the instrument. Would the indicated value of an unknown E_x be too high or too low? By what per cent?

5·5. From your general knowledge of the properties of mercury, what injurious action of mercury on a photovoltaic cell do you believe would be likely?

5·6. In terms of the electron gas theory of conduction devise an explanation for the existence of *contact potentials.*

5·7. Why are photovoltaic cells usually connected to comparatively *low* rather than high-resistance circuits?

5·8. Explain how it is possible to design a thermo-instrument with a vacuum thermocouple so that most of the scale is reasonably uniform or linear.

5·9. For what purposes are thermocouple instruments superior to other types?

PROBLEMS

5·1. The open-circuit potential difference of a given storage battery is 8 volts. The battery will maintain a current of 5 amp at a potential difference of 7.5 volts across the battery terminals.

(*a*) What is the internal resistance of the battery?

(*b*) Assuming the chemically generated emf of the battery to be the same, 8 volts, what will be its terminal voltage when charging at the rate of 2 amp?

5·2. A certain battery gives a potential difference of 10 volts across its terminals on open circuit. When supplying current to a resistance of 0.8 ohm the terminal voltage is only 8 volts.

(*a*) Determine the current flow and the internal resistance of the battery.

(*b*) What will the current flow be when the external resistance is 0.3 ohm and what will be the voltage across the battery terminals in this case?

(*c*) What would be the current through the battery if its terminals were short-circuited?

5·3. An automobile storage battery on open circuit has a terminal voltage of 6.6 volts. When momentarily short-circuited the current is 66 amp.

(*a*) What is the internal resistance of the battery?

(*b*) For what value of terminal voltage will the battery deliver maximum power? *Prove your answer.*

5·4. It is desired to provide means for repeatedly checking the temperature of a temporary furnace for annealing sheet steel punchings. The annealing temperature is approximately 800° C and critical to ±2° C. It is proposed to use an iron-constantan thermocouple with cold junction at 0° C, and a standard commercial potentiometer suitable for a range of 750° C to 850° C.

The potentiometer circuit is essentially as in Fig. 5·3 except that during measurement of the unknown E_x the galvanometer instead of connecting at *a* may be switched to any of fifteen other tap points along *abc* to provide a *coarse* adjustment of "slide-wire" voltage. The portion *cd* constitutes a *fine* adjustment and comprises eleven turns of slide wire in a helix on a 6 in. diameter drum. The switched steps are 5 ohms each and the actual slide wire is 0.5 ohm per turn. The instrument is designed to permit either 20 or 2 ma through the resistances *abcd*. (By adjusting *R*, it may be assumed here.)

(*a*) How many millivolts will be available at the temperature 800° C? (See Fig. 5·4.)

(*b*) Draw the complete circuit diagram of potentiometer and thermocouple as described (coarse adjustment for contact *c*).

(*c*) Determine which of the two permissible values of current should be used in the potentiometer.

(*d*) Compute the millivolt value for which the tap switch should be set.

(*e*) The helical slide wire is contacted by a steel-tipped spring attached to a bakelite drum which turns on a stationary vertical stud at the axis of the helix. The stud, being threaded for the same lead as the helix, permits the contact to follow the helix throughout its full length as the drum is revolved. The position of the contactor on the helix is indicated by two scales: (1) a vertical scale on glass in front of the drum, which permits the lower rim of the drum to register the number of turns, and (2) a circular scale on this rim which indicates the fraction of a turn.

Compute the number of Centigrade degrees represented by one turn of the helix and lay them off on a 6-in. circle (starting with zero) representing scale (2) above.

Compute the temperature corresponding to the lower end of the helix and lay off a linear scale, representing scale (1) above, of the temperatures corresponding to each successive turn of the drum.

5·5. A light meter with an overall accuracy of 5 per cent or better is required with range 0–75 foot-candles. A General Electric light cell is proposed together with a

D'Arsonval microammeter of suitable rating. From Fig. 5·6 it is noted that 75 foot-candles will produce up to about 240 μa, depending on the resistance connected across the cell. It is to be determined whether a standard 0–75-μa instrument can be used with a shunt of suitable size to enable the instrument to indicate foot-candles directly on its present scale, and to provide across the cell a value of resistance low enough to hold the error due to nonlinearity of microamperes versus foot-candles within bounds.

Four Weston instruments of the desired size (3 to 4 in. square), each with range 0–75 μa and 75 scale divisions, are available as follows in the order of price.

Model	*Scale Length (mm)*	*Ohms (approx.)*	*Accuracy*	*Price*
301	60	860	2%	$23.50
731	58.5	478	2%	24.50
801	80.3	450	2%	28.50
741	90	530	1%	35.50

(*a*) By comparative inspection eliminate two of the above instruments from further consideration.

(*b*) Determine the resistance of shunt required for each remaining instrument to indicate full scale for 75 foot-candles.

(*c*) For each assembly (from *b*) of microammeter, shunt, and light cell, determine at *mid-scale* the error incurred by assuming the foot-candles exactly proportional to the microamperes.

(*d*) Determine which instrument, if either, provides at mid-scale of foot-candles the desired overall accuracy of 5 per cent. The "accuracy" of the instrument is the maximum $\pm$ error of indication (here in microamperes or foot-candles) for any part of the scale, expressed as a percentage of the full scale. Note that although the instrument error is a $\pm$ quantity, the sign of the error produced by the non-linearity of cell response is not doubtful.

(*e*) Observe now that the error due to nonlinear cell response can be reduced at mid-scale by changing the values of the shunts found in (*b*) and transferring some of this error to the full-scale reading. By thus distributing the error throughout the scale, instead of concentrating it around mid-scale, a more balanced design is achieved. Estimate new values for the shunt resistors to accomplish this.

(*f*) On the basis of the foregoing, select the best microammeter for the job and indicate how well or badly it meets the desired specifications. Discuss here the susceptibility to temperature error which may be expected for the complete light meter of your selection.

Chapter VI

ELECTROMAGNETIC INDUCTION

6·1. The Phenomenon. Of the several sources of emf, electromagnetic induction has found the highest utility. We use this means for a great variety of tasks from the generation of all the huge amounts of energy in our power plants down to the minute amounts of energy in many of our microphones.

The facts of electromagnetic induction are not obtained from any source other than the physical evidence. Following the discovery by Oersted in 1820 of electromagnetic force as exhibited between a magnetic compass needle and a current-carrying conductor, it was felt by many that the essentially converse phenomenon of mechanical to electrical conversion should exist. It was not until 1831, however, that *Faraday* actually discovered the evidence. The remarkably complete series of experiments by which Faraday built up the epoch-making picture of electromagnetic induction is well known as one of the classical works of experimental physics. Every student of electrical phenomena may well take time to read the evidence as given in Faraday's "Diary" and to note the care with which he both observed and recorded the behavior of his experiments. The lesson may well be made more forcible by observing that our own *Joseph Henry* also discovered essentially the same facts as the English Faraday at so nearly the same time that later considerable controversy was incited as to who should receive priority credit. The work of Henry unfortunately suffers through lack of adequate documentary evidence, and the real truth may be forever unknown.

Without becoming involved in the complication of a thoroughly complete and inclusive statement which might cover all aspects of the phenomenon of electromagnetic induction of emf, we may first state the facts simply as follows. ***Whenever a magnetic flux links all or part of an electric conductor, any change in the amount of linked flux is attended by an emf in the conductor.***

The ***magnitude*** of the emf is directly proportional to the time rate of *change* in the amount of linked flux. Symbolically

$$e = \frac{d\phi}{dt} \qquad [6\cdot 1]$$

where $\frac{d\phi}{dt}$ = webers per second rate of change in linked flux.

e = volts emf induced in the conductor.

The ***polarity*** of the emf depends on the *direction* (clockwise or counterclockwise) by which the flux encircles or links the conductor, and on whether the *change* in the amount of linking flux is an increase or decrease.

It is best to represent all the above in the mathematical notation of *vector analysis*. In lieu of this it is essential that we employ a properly labeled sketch and invoke the aid of some kind of *right-hand rule* to assist our otherwise inadequate algebra.

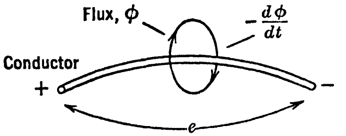


FIG. 6·1. Polarity of emf induced in conductor by decreasing linked magnetic flux.

The sketch of Fig. 6·1 represents a *decreasing flux* linking (encircling) a length of conductor in the direction of the arrows.

As applied to this phenomenon the right-hand rule is used to relate *direction of flux* and *polarity of emf*. The following statements are essentially the same but indicate the interchangeability of thumb and finger and relative foolproofness of their application.

(*a*) Grasping the conductor with the right hand so that fingers follow the direction of flux, the extended thumb points toward positive end of the conductor.

(*b*) Grasping a flux line with the right hand so that the extended thumb is in the direction of flux, the fingers point toward positive end of the conductor.

Observe that the flux lines are, as always, *closed* lines or loops but that the conductor is *not* required either to comprise a closed circuit or to carry current. Now note with care that the algebraically plus values of e and ϕ as sketched (and in accord with the right-hand rule) are shown by actual experiment to be *true only when the flux is* ***decreasing*** $(-d\phi/dt)$. Consequently we rewrite our basic relation algebraically as follows.

$$e = \mp \frac{d(\pm\phi)}{dt} \qquad [6\cdot 2]$$

Where, of course, the + and − respectively mean:

1. For $d\phi/dt$, increase and decrease of flux.
2. For ϕ, the sketched direction and the reverse.
3. For e, the sketched (right-hand) polarity and the reverse.

6·2. Classifications. The actual physical conformations of the elements of electromagnetic induction are indefinitely numerous. It is helpful, however, to classify them into three basic types according to the mechanism by which the necessary change in flux is produced, as follows.

1. By relative *motion* between flux and conductor.
2. By *change in permeance* of flux path.
3. By *change in mmf* (usually current) of the flux-producing agency.

Type 1 is illustrated by the examples of Fig. 6·2.

In Fig. 6·2*a* when the conductor (or magnet) is moved so as to *reduce* the flux linking with the conductor, emf is induced with polarity shown.

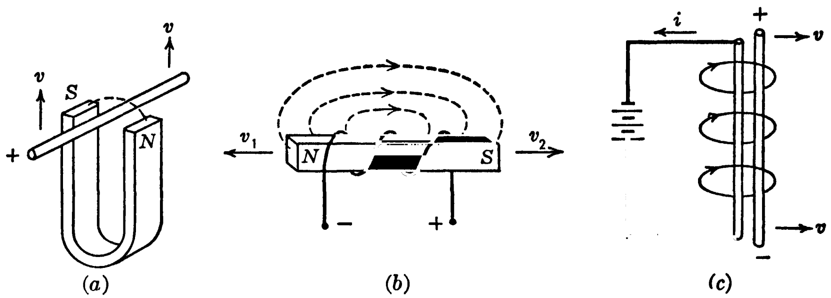


FIG. 6·2. Induction of emf by relative motion between conductor and linking magnetic flux.

Figure 6·2*b* suggests a bar magnet which when withdrawn from the coil in *either* direction v_1 or v_2 will give polarity shown. Figure 6·2*c* represents one conductor parallel to another which is carrying current. Movement of one conductor away from the other reduces the linking flux and induces emf of polarity shown. Electromagnets, of course, may be substituted for the permanent magnets in either Fig. 6·2*a* or 6·2*b*. Essentially the action of 6·2*a* is used for *magnetos* and of 6·2*c* for *power generators*. Type 2 is illustrated by the example of Fig. 6·3.

Figure 6·3 represents a horseshoe magnet NS with a coil wound on it. Movement of the soft iron piece *AA* in any direction away from the magnet reduces the permeance of the flux path, reduces the flux, and induces emf in the coil as shown. Flux *within* the magnet is, of course, from S to N. This action is used in *inductor type a-c generators* but they do not find extensive application.

Type 3 is illustrated by Fig. 6·4. In Fig. 6·4*a*, as switch *C* is opened and closed respectively by magnetic pull and spring return, the change in current *i* successively decreases and increases the flux and induces

emf e. No motion of the principal parts is required. This will be recognized as the familiar *induction coil.* As used in modern automobiles the switch C is mechanically actuated by a cam driven from the crank shaft. Figure 6·4b, of course, represents the *a-c transformer* which, with no moving parts, induces alternating emf in the upper coil (secondary) by reason of alternating flux from alternating current conducted through the lower coil (primary). It may be noted that Fig. 6·2c, if taken with conductors stationary and current varying (decreasing for polarity shown), illustrates an undesirable case where a telephone line on the same poles with a power line may have induced in it an unwanted emf by this same transformer action.

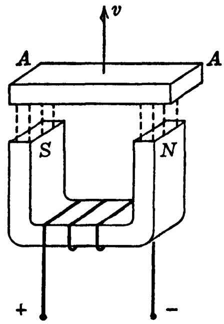
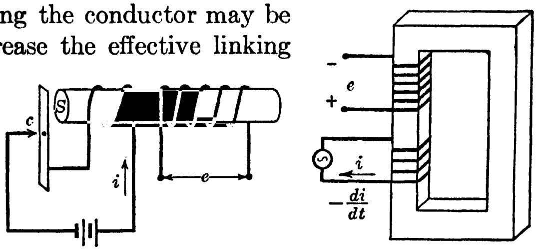


FIG. 6·3. Induction of emf by change in permeance of magnetic flux path.

6·3. Magnitude for the Coil. It frequently occurs that the conductor is in the form of a coil (Fig. 6·2b). The relation $e = d\phi/dt$ gives only the emf per turn of the coil. It will be observed that the device of coiling the conductor may be conceived *either* to increase the effective linking of the flux with the conductor *or* simply to connect in series the individual conductors, each comprising one turn, and by this means to sum the emf's induced in the respective turns. When the *same* change $d\phi/dt$ in linked flux occurs simultaneously in *all* N turns of a coil, the total emf induced is



(a) (b)

FIG. 6·4. (a) Induction coil. (b) Transformer.

$$e = N\frac{d\phi}{dt} \qquad [6\cdot3]$$

When, as in Fig. 6·2b, different amounts of flux ϕ' link the individual turns, it becomes necessary to sum either the emf's or the linking fluxes of the *individual turns.*

For the summation of turn emf's e' we write

$$e = \Sigma e' = \sum \frac{d\phi'}{dt} \qquad [6\cdot4]$$

It is usually better to employ the concept of summation of linking fluxes rather than the summation of turn emf's. This gives

$$e = \frac{d}{dt}(\Sigma\phi') = \frac{d\lambda}{dt} \tag{6·5}$$

where

$$\lambda = \Sigma\phi' \tag{6·6}$$

The term ***flux linkages*** with symbol λ and unit *weber-turns* is simply the algebraic sum of the fluxes linking the individual coil turns. When each turn is linked by the same flux we have

$$\lambda = N\phi \tag{6·7}$$

and equation 6·5 reduces to equation 6·3 as follows.

$$e = \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} \tag{6·8}$$

The more general form of the relation for induced emf given by equation 6·5 in terms of flux linkages λ will be encountered repeatedly and is worth mastering as an important basic concept. Note the important position occupied by λ in the diagram of Fig. 1·2. Its analogy with momentum M will be discussed in a later chapter.

6·4. Induced Current. It is to be noted explicitly that *current has had no place* in the foregoing. It is an *electromotive force* that is induced. There is nothing inherently different about an *induced* emf. Quite as a battery or other source of emf *may* or may *not* carry current, according to the nature of the circuit into which its emf is introduced, so an induced emf may or may not be accompanied by current. To speak of *induced current* is to employ a figure of speech which means only *current due to induced emf.*

6·5. A Special Case. In the construction of machines to execute mechanism of type 1 discussed in Art. 6·2 it is usual to design so that a section of straight conductor will change linkages with a flux which, in the vicinity of the conductor, is straight and perpendicular both to the conductor and to the direction of motion. As seen in Fig. 6·5 this gives rise to a concept of ***flux cutting*** which commonly is interchangeable with the concept of *changing linkage* and which is so useful that it merits special attention.

Figure 6·5 shows conductor of length l moving at velocity v perpendicular to and wholly within a flux of uniform density B. Movement

of the conductor through distance ds sweeps area $l\,ds$ and cuts (changes linking of) flux of amount $d\phi = B\,l\,ds$. It follows that

$$e = \frac{d\phi}{dt} = \frac{B\,l\,ds}{dt} = B\,l\frac{ds}{dt} = B\,l\,v \qquad [6\cdot 9]$$

If B = webers per square meter, l = meters, v = meters per second, then e = volts.

Relations of polarity and direction are easily indicated for this case by analogy with the *right-hand rule* used for the $F = Bli$ relation, which

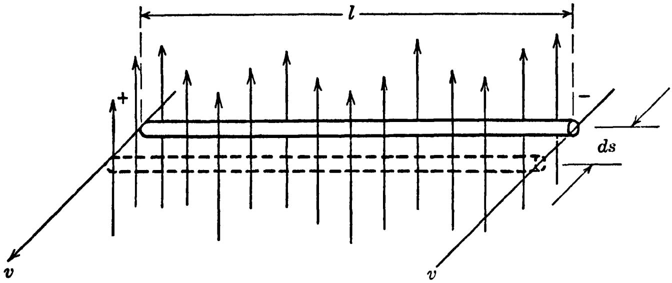


FIG. 6·5. Induction of emf by conductor cutting magnetic flux.

applies to the same physical picture (Fig. 6·5). The rule may be stated as follows.

The *middle finger*, perpendicular to palm and to forefinger and thumb, points to the + end of the conductor when the *forefinger* and *thumb* indicate direction of flux and motion of conductor respectively.

Observe that because all three factors are included in this hand rule, the earlier question of *in*creasing or *de*creasing flux is avoided and no minus sign is either desired or required in the $e = Blv$.

6·6. Energy Conversion. Let us now summarize for both emf and current the relations given by equations 6·9 and 3·3 respectively, transposed as follows.

$$\frac{e}{v} \leftrightharpoons Bl \rightleftharpoons \frac{F}{I} \qquad [6\cdot 10]$$

The dual equation shows that the conductor-in-flux (Bl) may either produce emf e due to velocity v, or force F due to current I. When *both* occur, the equation, eliminating Bl, gives the power relation

$$eI \rightleftharpoons p \rightleftharpoons Fv \qquad [6\cdot 11]$$

At any instant energy is being converted either from mechanical to electrical or vice versa, depending on the relation of I-direction to e-polarity and, likewise, by the relation of F-direction to v-direction in accord with the right-hand rules previously given for the e-v and F-I relations with Bl.

It is especially instructive to study the situation in Fig. 6·6. Here the emf E_g generated by the moving conductor is charging a battery which has voltage E_b and resistance R. Clearly, it is necessary that $E_g > E_b$ according to relation $E_g = IR + E_b$. Now if we reduce velocity v until $E_g < E_b$, the current must reverse to satisfy the voltage equation. From $F = BlI$ this also reverses F, and the energy conversion reverses, making the conductor-in-flux function as a motor instead of as a generator. At the same time the battery, of course, changes from charge to discharge.

Note carefully that the *direction of motion is* **not** *reversed* for this reversal of energy flow. On first thought this might seem to be unusual

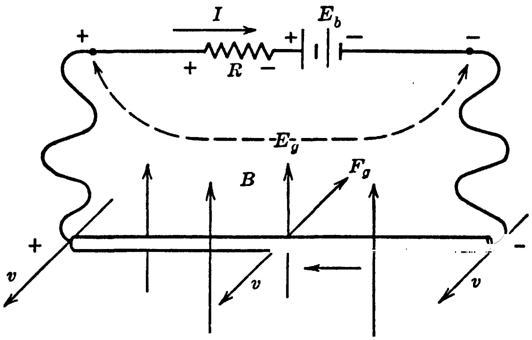


FIG. 6·6. An elementary generator converting from mechanical to electrical energy.

because we think of many mechanical experiences with springs, bodies moved against and with gravity, etc., where reversal of motion is required for reversal of energy flow. Not all mechanical experiences are of this kind, however; consider the motion of a hammer driving a nail into a wall. Energy is imparted to the hammer with the swing and then released on the strike without reversal of motion, except to repeat. The electrical phenomenon is unique, however, in that other prime movers such as water wheels, steam engines, and compressed air motors can reverse the energy flow only by reversal of motion (ruling out change in valve setting, of course).

6·7. The Faraday Disk or Eddy-Current Brake. An application of the electromagnetic induction of emf which was known even prior to the discovery of induction is embodied in the Faraday disk or eddy-current brake. This consists of a metallic disk arranged to rotate in a magnetic field as shown in Figs. 6·7*ab*.

The manner in which an emf is induced in the disk is best understood by conceiving the disk to constitute the limit reached when the spokes

of Fig. 6·7*c* are increased to infinite number. Each spoke while in the field has induced in it emf $e = \int Bv\,dl$ when B is the field density at a point distant l from the axis of rotation and moving with speed v. So long as the spokes are *separate* conductors the induced emf can cause no

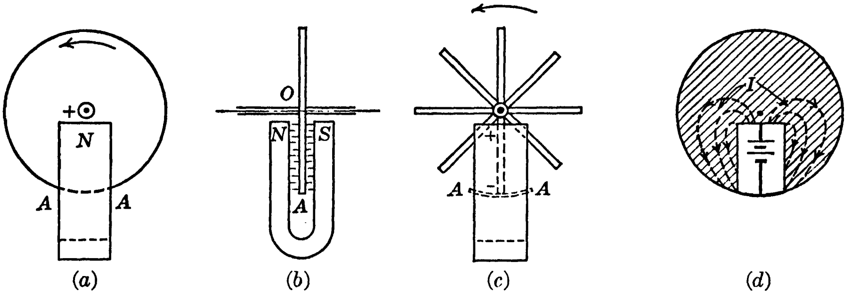


FIG. 6·7. Analysis of Faraday disk.

flow of current but, with their merger into a disk, paths other than radial are created which will carry current. The situation is much as though the flux-cut portion were cut out as in Fig. 6·7*d* and a battery emf substituted. The remaining portion of the disk in Fig. 6·7*d* clearly provides current paths which practically short-circuit the battery.

These currents, commonly called ***eddy currents,*** create in the disk both *I^2R heating* and electromagnetic *torque* (from $F = Bli$) which, for continued rotation of the disk, require expenditure of mechanical power equal to the I^2R heating of the disk. The conversion of mechanical

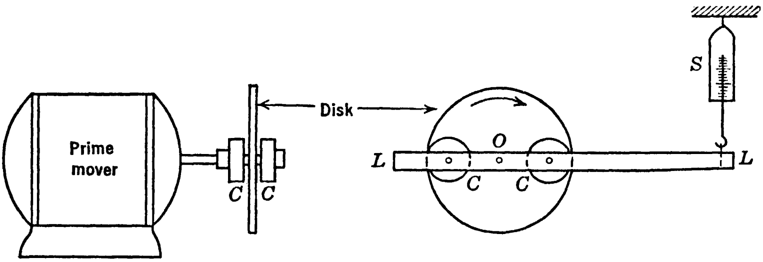


FIG. 6·8. An eddy-current brake.

energy into heat constitutes a braking action of practical consequence and gives rise to the term ***eddy-current brake.***

If the disk is mounted on the shaft of a prime mover, as in Fig. 6·8, while the magnet is mounted on a lever arm LL centered at the shaft O, the lever arm torque can be measured by scale S and the output of the prime mover computed. By using an electromagnet with coils $CCCC$ in

series with a rheostat, the field and torque can be *controlled* over any desired range. Such eddy-current brakes are commonly used in machine-testing laboratories for ratings up to a few horsepower in preference to mechanical brakes because of their smoother *performance* and *easier maintenance.* Beyond this range, dissipation of heat from the disk becomes a serious problem and ***electrodynamometers*** constructed along the lines of conventional generators are usual.

The eddy-current brake is widely used in *watthour-meters* to regulate the speed of the mechanism. For this application permanent magnets, adjustable as to position, are used and are readily visible in the newer glass-enclosed instruments for home, industry, or wherever electrical energy is purchased.

6·8. The Homopolar Generator. While some carelessness is found in the literature on the subject, it is clear that the *Faraday disk* or eddy-current brake would be wholly impractical as a source or generator of

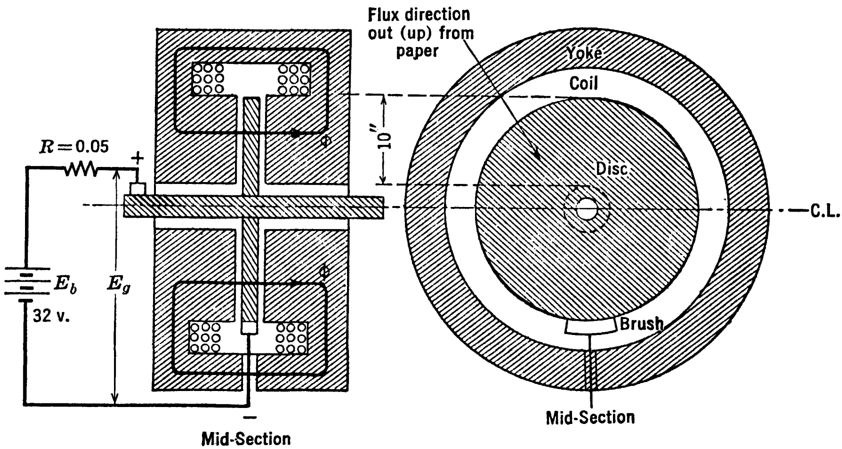


Fig. 6·9. A homopolar generator.

emf for engineering application. Both the power loss from the disk I^2R and the voltage drop from the disk IR are excessive beyond all reasonable tolerance. By a proper design, which recognizes the fundamental principles involved, it is possible to eliminate the eddy currents and obtain a more or less practical generator known as the homopolar generator.

To achieve this, ***each radius sector of the disk must simultaneously cut the same amount of magnetic flux.*** While the flux density need not be uniform along a *radius*, it must be uniform along the *circle* described by any given radius length within the disk. This is readily accomplished

by placing the flux-producing coil or ***field coil*** concentric with the disk as shown in Fig. 6·9.

Computation of the emf available from a homopolar generator with reasonable density of magnetic field and reasonable disk speed readily indicates that the emf is only of the order of a *few volts.* Some of these generators have been built with cylindrical rotors. A few homopolar generators have been built seriously for electroplating service, but the generator is not now commercially exploited. Aside from low voltage, the outstanding practical difficulty is the *high peripheral disk speed* which must be contacted by one of the brushes while carrying large currents. This is a fundamental requirement of construction which an amount of circumventive invention, rivaling that for perpetual motion, has failed to remove.

The application of the principle of electromagnetic induction of emf to the practical generators commonly in use involves a treatment extensive beyond the limitations of this work and is available in many excellent books of various thoroughness. The homopolar generator is considered here as an example where simplicity is not synonymous with practicality.

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*7. Turner, "Makers of Science—Electricity and Magnetism," Oxford, pp. 88–95.

QUESTIONS

6·1. Explain with appropriate sketches how the concept of *flux cutting* is equivalent to the changing of flux linkages.

6·2. Expound the true meaning of the expression *induced current* and the erroneous inference which it invites.

6·3. It frequently happens that all the linking flux does not link all the turns of a coiled conductor. Explain what procedure would be necessary to compute the emf induced in the entire coil.

6·4. Plan and describe in detail an experimental procedure by which the facts pertinent to the determination of the polarity of an induced emf could be established.

6·5. Label a sketch similar to Fig. 6·6 with the proper polarities of emf's and directions of current, velocity, force, and flux to show conversion of energy from electrical to mechanical.

* Especially recommended.

6·6. Describe a homopolar generator (see ref. 3, 4, or 5) and explain how it differs from the Faraday disk or eddy-current brake.

6·7. Determine the polarity of O and A in Fig. 6·7*a*.

6·8. Study the homopolar generator and determine what mechanical and electrical difficulties prevent it from attaining commercial importance.

PROBLEMS

6·1. Given a conductor as in Fig. 6·6 of the text,

Let $l = 10$ cm.
$B = 15 \times 10^{-5}$ webers per cm^2.
$E_g = 16$ volts.
$R = 0.4$ ohm.
$E_t = 15$ volts.

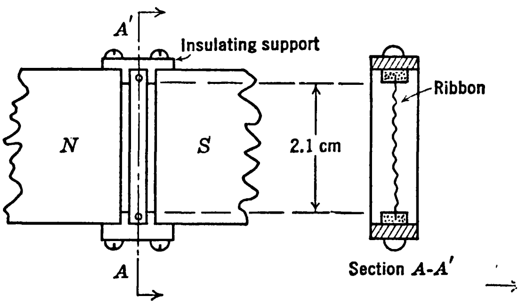
Compute:

(*a*) The mechanical power in foot-pounds per second required to move the conductor relative to the field.

(*b*) The velocity of the conductor (normal to the field).

(*c*) The pounds force required to move the conductor.

6·2. The accompanying sketch represents the essential construction of a ribbon or "velocity" microphone. The active element is a thin metallic crimped ribbon



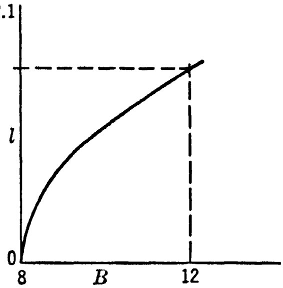
stretched within (and perpendicular to) a uniform magnetic field so that sound waves cause it to cut the field and induce an emf proportional to the ribbon velocity at each instant.

(*a*) It is desired to compute the emf induced between the ribbon terminals for one particular instant during a sustained simple vibration. The following conditions are assumed to be fulfilled by the microphone.

(1) Uniform flux density $B = 10$ kilolines per centimeter^2.
(2) Length of ribbon before crimped $= 3.8$ cm.
(3) Length of crimped ribbon $= 2.1$ cm.
(4) Ribbon moves as a rigid body.
(5) Ribbon velocity at given instant $= 50$ cm per second.
(6) Ribbon carries negligible current.

(*b*) Solve the above for a condition where poor workmanship results in a nonuniform air gap such that the flux density varies parabolically along the ribbon from

$B = 12$ kilolines per centimeter^2 at the upper end of the ribbon to 8 kilolines per centimeter^2 at the lower end of the ribbon as shown in the graph.



6·3. Given a homopolar dynamo, as in Fig. 6·9, with the following data.
Battery voltage (constant) $E_b = 32$ volts.
Circuit resistance, $R = 0.05$ ohm.
Radius of disk $= 12$ in., portion of radius in the flux $= 10$ in. (see Fig. 6·9).
Flux density (uniform) $B = 120 \times 10^{-6}$ webers per inch.^{2}

(*a*) Find:

(1) Direction of rotation of disk for indicated polarity of E_g.
(2) Rpm S of disk for $E_g = E_b$.
(3) Equation for $S = f(I)$, assuming $E_g > E_b$ or dynamo acting as *source* of electrical energy.
(4) Equation for $T_g = f(I)$.
(5) Equation for $S = f(T_g)$.

(*b*) Let the friction of the generator be expressed by torque $T_f = S/2000$ lb-ft. The generator is driven by a *direct-coupled* prime mover which delivers torque $T_m = 26.80 - (S/300)$ lb-ft. Compute:

(1) Rpm S of the prime mover and generator.
(2) Watts power delivered to the battery (net value put in storage).

CHAPTER VII

ALTERNATING EMF

7·1. An Experiment. After observing how an emf is induced in an elemental conductor simply by moving it within and normal to a magnetic field (Fig. 6·5) let us proceed to determine the effect of moving the conductor with a readily produced vibrating motion normal to the field.

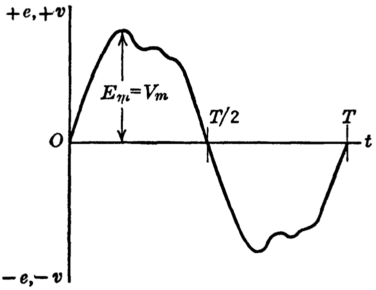


FIG. 7·1.

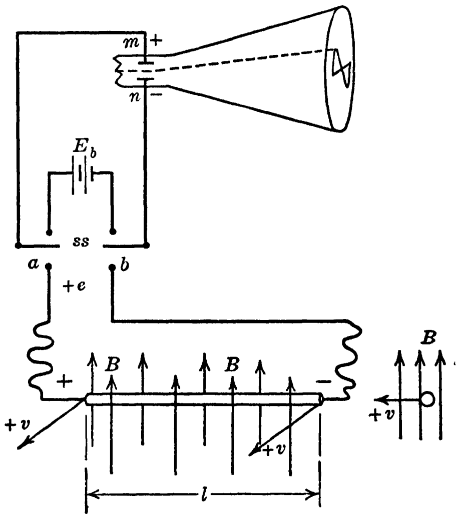


FIG. 7·2.

Polarity of alternating emf generated by conductor vibrating across a magnetic field.

From $e = Blv$ it is clear that, with B and l constant, e is simply proportional to v. Now let the conductor vibrate in the above manner and according to the pattern represented by $v = f(t)$ as graphed in Fig. 7·1.

Because e is proportional to v at each instant, the $e = f(t)$ graph is similar to the $v = f(t)$ graph and can be identically the same curve if the appropriate scale for e is used as in Fig. 7·1.

The emf with periodically reversing polarity which we have obtained is called an ***alternating emf*** or more commonly an a-c emf (alternating-current emf) although the term is literally nonsensical.

7·2. Importance of Alternating Current. The importance today of alternating current as compared with direct current hardly needs emphasis. Practically all electrical energy for power purposes is now generated as alternating current even for the minority of applications where it will be converted to direct current before use. The electrochemical industry is today the major consumer requiring d-c power. There are several reasons for the predominance of alternating current which will focus more sharply as study proceeds beyond this work. Outstanding among these are the relative simplicity of a-c generators and motors together with the ease of conversion from one voltage to another by motionless equipment (transformers) so that the wide range of optimum values required for generation, transmission, and application can be fully satisfied with practical over-all economy and reliability.

In the fields of communication and electronics it is clear that dealing with electrical replica of speech vibrations involves alternating current of highly complex patterns and rigorous *fidelity* requirements. At the same time it must not be overlooked that the apparatus in these fields demands supplies of direct current with a uniformity or constancy normally unequaled in the requirements of the power field.

7·3. Algebra versus Polarity. By following the conventional significance of the algebraic plus and minus signs there need be no confusion about the physical meaning of the graph of Fig. 7·1 with respect to the apparatus of Fig. 7·2. Clearly the velocity values graphed in the $+v$ region signify that the direction of motion in Fig. 7·2 is *actually* that designated by the arrows labeled $+v$, while the velocity values graphed in the $-v$ region signify actual direction of motion *opposite* to the arrow direction labeled $+v$ in Fig. 7·2.

With this in mind we may well appreciate that the use of the polarity labels + and − (positive and negative) apply to the emf of the apparatus in Fig. 7·2 in the same algebraic sense as do the arrows for the velocity. We stress this point because the question so often is raised: "How can + and − polarity labels make sense when the emf is *alternating* and therefore no more +− than it is −+? Furthermore a-c voltmeters recognize no polarity—*does alternating emf really have polarity?*"

If we do not feel altogether confident in the above algebraic notation for interpreting the significance of a-c polarity, let us turn to purely physical considerations. We shall establish a physical instrument to interpret polarity, a basic standard to which we may turn in time of doubt and recalibrate the meaning of our symbolism. The oscillograph is eminently suited to this purpose. In Fig. 7·2 an electrostatically controlled oscillograph is shown ready to connect to the emf of the vibrating conductor and produce an oscillogram of $e = f(t)$.

Let us first determine by experiment which control terminal, m or n, must be *positive* to deflect the electron beam *above* the zero axis and into the algebraically *plus* region of the screen or oscillogram. Moving switches ss upward momentarily, we connect a battery, or other d-c source (E_b) with readily identified polarity, to the oscillograph and observe the trace on the screen. In Fig. 7·2 we find that the polarity shown deflects the beam upward or algebraically plusward and accordingly we *label m positive and n negative.* We have thus calibrated the polarity of the oscillograph for algebraic plus and it is ready for use.

We now move switches ss downward, obtain the oscillogram of Fig. 7·1, and remove the oscillograph connections. What does the oscillogram now indicate about the polarity of e taken between the ends of the vibrating conductor? In an absolute sense it means nothing unless, on the points a and b of the circuit, we have left an imprinted record of the + and − of the oscillograph leads which we connected there in taking the oscillogram. Given these identifying labels, we are now prepared to assert that, during the *above*-axis (plus) trace on the oscillogram the emf generated by the vibrating conductor had the polarity which we have just *marked* on it, and that during the *below*-axis (minus) trace it had the *reverse* polarity from that marked on it. By thus providing the necessary +− tags on the *circuit* the meaning of the oscillogram becomes unambiguous. ***An oscillogram lacking either circuit diagram or identifying labels is seldom useful.***

7·4. Double Subscripts. We are usually interested in more than one emf in a circuit and it becomes necessary to further identify each oscillogram with the circuit points from which it was obtained.

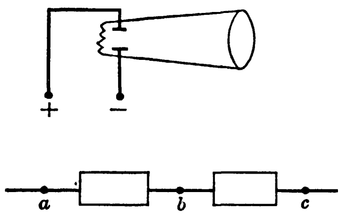


FIG. 7·3.

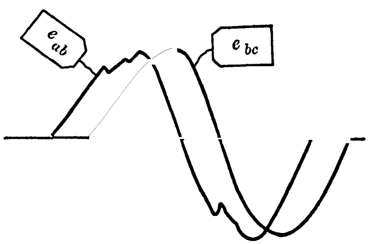


FIG. 7·4.

We then label the circuit points abc, etc., as in Fig. 7·3 and *tag each oscillogram* with appropriately subscripted labels e_{ab}, e_{bc}, etc., as in Fig. 7·4. In order to avoid more than the essential tagging of the *circuit* we shall delegate to the subscripts a second task, by assigning a significance to their *sequence.* ***The circuit point designated by the first***

subscript shall be that to which the positive (+) lead of the oscillograph was connected.

It follows that $e_{ba} = -e_{ab}$, $e_{cb} = -e_{bc}$, etc. We now need leave no + and − imprints of the oscillograph leads on the circuit so long as we agree to abide by this arbitrary significance of subscript sequence.

Unfortunately this is not a universal agreement. In fact it will be found that much confusion exists in the field as to the specific meaning of the whole double-subscript nomenclature although it is widely used and generally accepted to occupy a place of major importance in the symbolism of electrical engineering. Because eventually it will be found imperative to attain proficiency in the use of double subscripts, the reader who achieves the advantage of a thorough and early understanding of them will find his preparedness well repaid.

7·5. A-C Terminology. There are several terms peculiar to the periodic quantities of alternating current which it is important to define and add to our vocabulary before proceeding farther. These are as follows.

Oscillating Alternately increasing and decreasing in value, always within finite limits.

Periodic Oscillating with all values recurring, for equal increments of the independent variable.

Pulsating Periodic with values all positive or all negative.

Alternating Periodic with values alternately positive and negative (usually symmetrical, i.e., the positive and negative graphically superimposable).

Cycle The minimum series of values which, repeated, describe the periodic quantity (symbol ~).

Alternation (positive or negative) One complete set of the positive, or of the negative, values which comprise one cycle.

Period For a periodic quantity the smallest value of the increment of the independent variable which separates values recurring with the same rate and direction of change.

Time period The time required to execute one cycle of a periodic function of time (symbol T).

Frequency Reciprocal of the time period; cycles per unit time (symbol f). Usually referring to the propagation of a disturbance but used here for the graphic representation of the cycle in Cartesian coordinates.

Wave form Shape of the wave, i.e., peaked, triangular, square, sawtooth, sinusoidal.

Amplitude Maximum magnitude of an alternating quantity.

Phase For a particular value of a periodic quantity, the fractional part of a period or cycle through which the quantity has advanced measured from a chosen origin.

Phase angle The angle equivalent of the phase in radians or degrees based upon the cycle or period as 2π radians or 360 degrees.

Phase difference The phase difference between two periodic quantities which have the same period and wave form is the fractional part of a period (not greater than one-half) through which the one wave would have to be advanced with respect to the other in order that when plotted in percentage of amplitude they may coincide.

Angular phase difference The angle equivalent of the phase difference (same basis as phase angle versus phase).

In phase Zero angular phase difference.

Phase quadrature Angular phase difference of 90 degrees, $\pi/2$ radians or ¼ cycle.

Phase opposition Angular phase difference of 180 degrees, π radians or ½ cycle.

Phase sequence When two or more periodic quantities such as A, B, C have the same period and wave form but are not in phase, the sequence with which they reach corresponding phases of their cycles is termed their phase sequence. When B follows A, and C follows B the sequence of course is ABC.

For a complete list of definitions the reader should consult the recently published (1942) 300 page American Standard "Definitions of Electrical Terms" by the A.I.E.E. This has been in tentative form for several years and is now an approved ASA Standard.

7·6. Sinusoidal Emf. There are numerous mechanisms by which the vibration of the conductor of Fig. 7·2 may be executed. Possibly the simplest would be to provide an elastic restraint (spring) and utilize mechanical resonance. It would then tend to pursue the simple harmonic motion which so often occurs in nature and which we know can be represented as a sine function of time, $v = f(t)$, by simple mathematics:

$$v = V_m \sin \omega t = V_m \sin 2\pi \,\mathrm{ft}$$

where v = velocity at any instant of time t.

V_m = maximum velocity.

ω = angular velocity (constant) in radians per second of a real or equivalent rotating source of the motion.

f = frequency or time rate of vibration in cycles per second.

Physically, the rotation implied by ω is conveniently represented by a crankpin rotating about a horizontal axis at constant rpm. The vertical component of motion of the pin is, of course, simple harmonic motion. We shall here consider only vibration of constant amplitude V_m = constant.

Since $e = Blv$, the emf induced in the sinusoidally vibrating conductor is, of course, like the conductor velocity, a sine function of time

$$e = E_m \sin \omega t$$

or

$$e = E_m \sin 2\pi \mathrm{f} t$$

where e = induced emf at any instant of time t.
E_m = maximum value or *amplitude* of emf.
ω and f are as before.

An oscillogram of e versus t is shown in Fig. 7·5 for one cycle or one period. Because it will soon be found necessary to attain some proficiency in sketching an acceptable facsimile of the sine wave, it is well to observe some graphical "aids to navigation." Note in Fig. 7·5 that $e = E_m/2$ when $t = T/12$, and that $e = 0.866E_m$ when $t = T/6$.

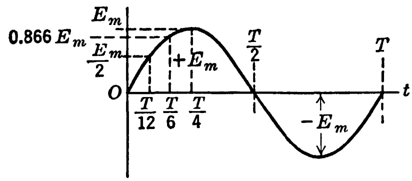


FIG. 7·5. Oscillogram of a sinusoidal emf.

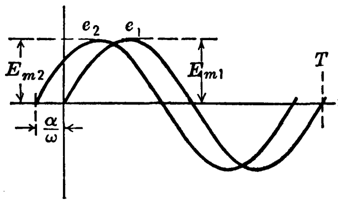


FIG. 7·6. Phase advance of sinusoidal emf by angle α.

It does not always suffice to represent the sinusoidal emf, with $e = 0$ at $t = 0$. When a second emf e_2 not in phase with the first e_1 is to be considered, the angle of phase difference must be accounted for in the equations of emf versus t. Let the oscillogram be as in Fig. 7·6. Here the corresponding phases of e_2 occur *prior* to those of e_1. Therefore the emf e_2 is said to be *leading* e_1 by the phase angle α. The equation for e_2 is

$$e_2 = E_{m2} \sin (\omega t + \alpha)$$

It is well to study this generalized sine function in relation to the oscillogram and both reconcile and remember that ***plus** angle* $(+\alpha)$ ***advances** the wave.* The wave of course is retarded by $-\alpha$. On an axis of time the phase angle, as shown in Fig. 7·6, must be expressed as α/ω to be dimensionally consistent.

7·7. A Phase Angle Convention. When the angle α is such that $180° < \alpha < 360°$ it is customary to use $\beta = 360° - \alpha$ as the phase angle instead of α. When this is done it must be observed that e_2 is *lagging* e_1 by angle β instead of *leading* by angle α.

This may at first seem to be taking unwarranted liberty with the description of the physical sense of the phenomena represented by e_1 and e_2. Surely when a horse lags so far behind on a circular race track that, to a late observer, he might appear to be leading the race, it would be contrary to fact to state that he is leading. For the electric circuit we are *not at present concerned with the starting of the race* and are content to share the illusion of the late observer.

Time $t = 0$ then does not signify the beginning of the phenomena; it signifies the *beginning of observation.* Eventually we shall be concerned with the actual initiation of the phenomena for some simple cases. We appropriately term this period of getting under way the ***transient*** period in distinction from the ***steady state*** condition with which we are now concerned. In this sense we speak of transient emf's and transient currents or simply transients in distinction from steady state emf's, etc.

7·8. Sinusoidal Flux Distribution. We have observed how a sinusoidal emf can be generated from $e = Blv$ by imparting simple harmonic motion $v = V_m \sin \omega t$ to a conductor of length l wholly within and normal to a magnetic field of uniform density B. It is fairly obvious mathematically that we might also produce sinusoidal emf by making v = constant and varying B sinusoidally according to $B = B_m \sin \omega t$. This is essentially the method used by all standard rotating a-c generators or ***alternators,*** as they are commonly called. Let us examine the physical execution of the method.

Figure 7·7 represents one of a number of conductors located on the periphery of the *armature* which for this design is the rotor of the

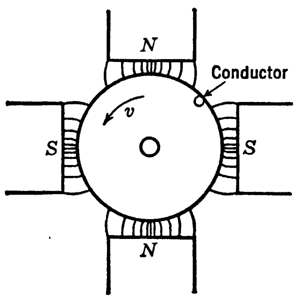


FIG. 7·7.

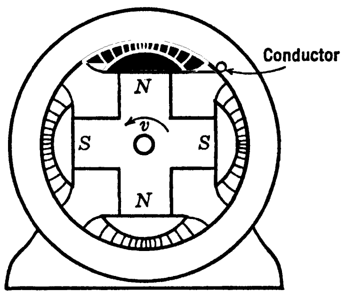


FIG. 7·8.

Distribution of magnetic flux density in alternators for sinusoidal emf.

machine. The magnetic design is such that all the flux is not only radial but also so distributed that the density B encountered by the conductor will vary sinusoidally as the rotor revolves. So long as the peripheral speed of the conductor is constant, the velocity of the

conductor with respect to the radial flux will be constant ($v = V$) and the induced emf will be sinusoidal according to the relation

$$e = Blv = lVB_m \sin \omega t = E_m \sin \omega t$$

Modern alternators are built, as in Fig. 7·8, with the conductors (the armature winding) embedded in the stationary *stator*, and the flux-producing member or *field structure* becomes the rotor. This design avoids using rubbing contacts or *slip rings* and *brushes* to conduct the high voltages (2300 to 13,800) and large currents (thousands of amperes) which these machines carry.

Possibly it is easier to see in Fig. 7·8 than in Fig. 7·7 how a sinusoidal *space* distribution of flux around the periphery of the rotor results in a sinusoidal *time* variation of flux density at the *conductor* when the rotor is revolving.

7·9. Motionless and Motional Emf. When the flux is produced by a controllable electric current, as in an electromagnet, the flux may be

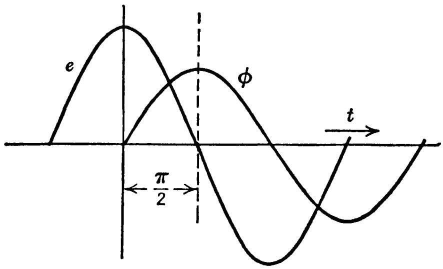


FIG. 7·9. Sinusoidal emf induced by sinusoidally varying magnetic flux.

varied sinusoidally without mechanical motion. In this case where $v = 0$ the $e = Blv$ is useless but the original $e = N(d\phi/dt)$ shows that when $\phi = \phi_m \sin \omega t$ the emf is $e = N(\omega\phi_m \cos \omega t) = \omega N\phi_m \cos \omega t$ or $e = E_m \cos \omega t$.

The cosine function, it must be recalled, is graphically identical with the sine function except that it is $+1$ at zero angle instead of at 90 degrees. In other words the cosine function is in leading (phase) quadrature with the sine function or is $90° = \pi/2$ radians ahead of the sine function. As time t elapses (increases), any given value of the cosine occurs 90 degrees in *advance* of the corresponding value of the sine. This is expressed by $\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$ where $+\frac{\pi}{2}$ is, of course, a phase angle of advance. Thus it follows that $e = E_m \cos \omega t = E_m \sin\left(\omega t + \frac{\pi}{2}\right)$ as graphed in Fig. 7·9.

This form of induction, as before mentioned, is basic for *transformer action* as exemplified in induction coils, induction regulators, transformers, and the undesired *inductive interference* between telephone and power lines. To distinguish between these ***emf's due to transformer action*** and the previously discussed emf's induced by reason of actual mechanical motion of magnetic poles or conductors, the latter are commonly identified as ***motional emf's.*** Analysis of some machines involves recognition of both kinds of emf in the same machine.

7·10. Current. Sinusoidal emf's like other emf's of course may exist without an attendant current. When currents do exist they may constitute simple harmonic motion along the conductor and if so are representable as sine functions in the form $i = I_m \sin(\omega t + \alpha)$. Further consideration of both sinusoidal currents and voltages as well as the attendant power and energy will be pursued at length later in this work.

7·11. Why Sine Waves? From the foregoing it would seem that the mathematical treatment of alternating emf's and currents is considerably simplified if we can restrict our interest to *sinusoidal* emf's and currents. It may be noted that we went to some pains to design the generators of Figs. 7·7 and 7·8 so that *sinusoidal* emf's would be produced. Can it be that this is done merely to simplify the mathematical treatment? Be assured that this is not the reason; there are more fundamental motives dictated by physical aspects of electric circuit phenomena. We are not in position to deal with these until we have studied inductance and capacitance. There is another aspect of the matter however which we should be prepared to consider at once.

By both mathematical and physical means we find that a-c waves of any shape encountered in alternating emf's and currents may be resolved into a series of sinusoidal components of appropriate frequency, amplitude, and phase. These components are known as ***harmonics*** and have frequencies which are multiples of the lowest or ***fundamental*** frequency. Even for the most difficult wave shapes, such as the rectangular, a mere dozen or so harmonics suffices for nearly all practical requirements. The *Fourier analysis* offers a combination analytical and graphical means for determining the component waves, while several instruments known as *wave analyzers* perform the analysis by physical means when connected to the line emf or current in question. The instruments vary widely in the basic principles employed and some of them are of entirely practical and successful design.

Because a knowledge of sinusoidal emf's and currents can by this means be applied to waves of any shape, it becomes clear that we are by no means neglecting consideration of a-c waves in general when we concentrate on the sine wave. It should be quickly added, however,

that not *all* problems involving non-sine waves are best handled by analysis into sine components. No generalized criteria for recognizing such cases seem feasible.

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2. LAWRENCE, "Principles of Alternating Currents," McGraw-Hill Book Co., Second Edition, 1935, pp. 29–35.
3. KARAPETOFF, "The Electric Circuit," McGraw-Hill Book Co., Chap. IV, Art. 12, pp. 31–34.

QUESTIONS

7·1. Define:

(*a*) Alternating	(*d*) Period	(*g*) Wave shape
(*b*) Alternation	(*e*) Cycle	(*h*) Amplitude
(*c*) Alternator	(*f*) Frequency	(*i*) Phase angle

7·2. Derive for the alternator, a relation among rpm, poles, and frequency. See Figs. 7·7 and 7·8.

7·3. What facts about a *sine* voltage (or current) are required to describe it *completely?*

7·4. Since an alternating voltage has no *fixed* polarity, what is the meaning of polarity signs (+ and −) on an a-c circuit diagram?

7·5. Express $E_m \cos \omega t$ as a sine function.

7·6. Referring to the expression $i = I_m \sin (\omega t + \alpha)$ sketch the wave and indicate clearly the effect of α for both positive and negative values of α.

7·7. Explain what is meant by motional emf's.

7·8. Cite a case of electromagnetic induction where the relation $e = Blv$ is useless to determine the voltage e.

PROBLEMS

7·1. A sinusoidal alternating voltage rises from zero to one-half maximum value ($E_m = 100$ volts) in 0.005 second.

(*a*) What is the time period of the emf?

(*b*) What is the frequency of the emf?

(*c*) Write the equation for the emf in terms of the determined constants.

7·2. A sinusoidal emf has an instantaneous value of +50 volts at one-half of an alternation after passing through zero. What is its instantaneous value two-thirds of a cycle after passing through zero?

7·3. (*a*) Write the equation of a 50-cycle sinusoidal alternating emf whose maximum value is 110 volts.

(*b*) What is the instantaneous value of this voltage 0.01 sec after it reaches a positive maximum value?

(*c*) Repeat (*b*) for $t = 0.0083$ sec after voltage passes through zero with positive slope.

7·4. The voltage induced in a six-pole a-c generator varies with time according to the sine law and has a maximum value of 100 volts.

(*a*) Plot the curve of instantaneous values of e for at least one cycle.

(*b*) Mark the scale of abscissas in fractions of a cycle, and in radians.

(c) If the generator runs at 1000 rpm, what is the time period of the emf? Add scale of time to the abscissas.

7·5. When $t = 0$, a sinusoidal emf has a value $e = +50v$ and is increasing to a maximum which it will reach in 0.01 sec. When $t = 0.03$ sec, $e = -50$ volts.

(a) Compute the amplitude E_m and frequency f of the emf.

(b) Write the numerical equation for e versus t.

(c) Sketch the oscillogram of e versus t.

CHAPTER VIII

ELECTRIC POWER CIRCUITS

8·1. Electrical Distribution. The conveyance of electric energy by conduction is termed "transmission" or distribution. ***Transmission*** refers to long distance *express* lines—the wholesale carriers. ***Distribution*** usually refers to comparatively short *local* lines—the retail carriers. In America nearly all transmission and most distribution lines carry alternating rather than direct current. This is due principally to the ease with which alternating voltage can be stepped up or down to meet the various needs of the individual members which comprise the entire system or *network.*

Since the successful introduction of the mercury-arc rectifier in this country in large units, considerable research has been directed toward d-c transmission by the process of generating alternating current, rectifying to direct current for transmission, and then *inverting* back to alternating current for flexible distribution. It is entirely possible that d-c transmission may come to be a sizeable reality. Whether or not this develops is unimportant for the immediate purpose because the principles to be considered here are largely applicable to either alternating or direct current.

There are two basic distribution circuits:

(*a*) The constant current or *series* system.

(*b*) The constant potential or *parallel* system.

The series circuit or Thury system, as it is known in Europe, is little used in America and only for street lighting. The connection simply consists in linking all apparatus in a single loop or endless chain (like so much link sausage). It is unpopular for general use because of its unsafe voltages and its comparative inflexibility for power service.

The parallel or constant potential circuit is a ladder-like arrangement in which the load and source "rungs" are connected by the conductor "sides." The *constant* potential is an ideal condition which, so long as conductors will not carry current without an IR drop in potential, can only be achieved approximately. A limit of 3 per cent voltage variation is usually specified for lighting circuits—more for power circuits.

Four requirements must be considered in determining the size of wire to be used for a given circuit.

1. Thermal (no fires).
2. Mechanical (no breaks).
3. Electrical (no low-voltage complaints).
4. Economical (no bankruptcy).

Let us consider these in the order named.

8·2. Thermal Requirements. In 1897 a National Electrical Code (NEC) was originated through the efforts of several interests, particularly the National Board of Fire Underwriters (1866). The present revision is an American Standard, approved by the American Standards Association. The rules of the NEC are commonly included in the local interior wiring ordinances or the *building code* of each city or town and are enforced through the local inspector who must approve each installation before its use is permitted. A table of allowable current capacities for various insulating materials and sizes of wire is provided by the NEC and is reproduced in all electrical handbooks as well as many other publications. These provide a liberal margin of safety for the temperature rise of the various sizes so that not only immediate fire hazard but also eventual danger from too rapid *deterioration of insulation* may be avoided. It may be observed that for different sizes of wire the surface for *heat dissipation* is *not* proportional to the wire section which determines the *heat production.* Consequently the allowable current is *less* per circular mil as the wire section is increased.

8·3. Mechanical Requirements. Detailed specification of sizes for mechanical strength of interior wiring is unnecessary. The NEC does require "*no wire smaller than No.* 14 *for interior wiring*" and nothing smaller than No. 18 for the wiring of the lighting fixtures. The No. 14 is specified, not altogether from the mechanical viewpoint, however, because 15-amp fuses are usual and No. 14 is the smallest permissible for carrying 15 amp. Other mechanical considerations than size of conductor are numerous beyond consideration here.

The longer *outdoor* lines, especially in long-distance transmission, require very careful consideration of mechanical adequacy, and are commonly computed for stresses due to loading by wind and ice. The current-carrying wires are sometimes supported by ***messenger cable*** or ***span wire*** provided solely for this mechanical duty. Other designs employ a *steel core.* The span wire is a necessity for first-class overhead trolley-wire installations where sag in the conductor is of course not permissible and the *catenary construction* is commonly used.

8·4. Electrical Requirements. The most difficult problem of electrical distribution is maintenance of the desired voltage when, as usual, the load is not constant. The obvious requirement is wire sufficiently large to keep the unavoidable voltage drop within the desired limit. It is sometimes possible to use a higher voltage or a modified circuit which will avoid the obvious but expensive alternative of larger wire size.

If acceptable for safety and for load requirements, higher voltage for a given kilowatt load proportionately reduces the current and IR drop in a given line. This may be shown as follows. Given

E_1 = generator or sending-end voltage.
E_2 = load-end voltage.
I = load and line current.
$P_2 = E_2I$ = power taken by load.
R_L = line resistance.
$P_L = I^2R_L$ = power lost in line.

If we now consider the load power P_2 and the line resistance R_L to be *fixed*, the per cent IR_L line drop in terms of E_2 will be

$$\frac{IR_L}{E_2} = \frac{P_2}{E_2}\frac{R_L}{E_2} = \frac{P_2R_L}{E_2^2} = \frac{K}{E_2^2}$$

Thus the per cent line drop for a given line resistance and load power is ***inversely proportional to the square*** of the load voltage, and the reason for *high-voltage* transmission is evident. Furthermore, the relative ease with which a-c voltages may be transformed to higher or lower values explains why they are preferred to d-c voltages for general use in the conveyance of electrical energy; the voltage may be changed in going from one part of the system to another in order to maintain the optimum efficiency compatible with the diverse requirements for safe potentials which apply in the various parts of the system.

Several modifications of the simple two-wire circuit are in general use, as follows:

(*a*) The three-wire Edison system.
(*b*) The loop system.
(*c*) The radial system.
(*d*) The network.

8·5. The Three-Wire Edison System. This system is used extensively for distribution to homes and other small consumers of electric power (Fig. 8·1). While some advantage is found in thus making available both 230 volts and 115 volts, the major consideration is the

reduction in copper possible for a given voltage drop. When the loads A and B are equal, current I_0 is zero, and the circuit is essentially a 230-volt circuit with the load constituted by A and B in series.

It is intended in the design of the circuit that loads A and B shall be balanced (equal), normally, and that the neutral wire shall serve mainly during load unbalance to maintain the desired *voltage* balance across A and B and to conduct the consequent current I_0. It is important that the *neutral wire be connected solidly,* with *no fuse,* because an open neutral would open the possibility that one load A or B might

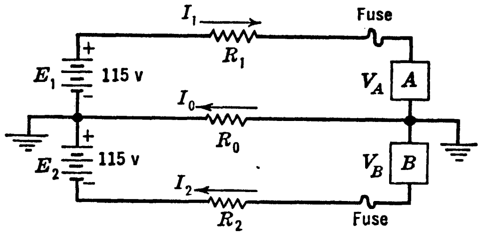

FIG. 8·1. A three-wire Edison circuit.

receive 230 volts instead of the expected 115. In such event both consumers A and B would have just cause for complaint.

By reason of various abnormal conditions such as possible contact with another and higher-voltage line, abnormal voltages may appear on a line. If some one wire of the line is not rather definitely connected to earth or *grounded* there is considerable danger that a consumer or his equipment may receive from line to ground a magnitude of potential far greater than any which the rating of the line would lead him to expect. Consequently it is today considered an important safety measure to make certain that one wire of a line is well or solidly grounded. Of course this grounded wire must not be fused. For this and other reasons it is appropriate that in a three-wire system *the neutral should be the grounded wire* as in Fig. 8·1.

To one whose experience so far may have been confined mostly to two-wire lines the three-wire line may sometimes appear to behave unconventionally. Referring to Fig. 8·1 let us consider an example where, in addition to the values shown,

$$R_1 = R_0 = R_2 = 0.20 \text{ ohm}$$

$$I_1 = 50 \text{ amp}$$

$$I_2 = 40 \text{ amp}$$

It follows that

$$I_0 = I_1 - I_2 = 50 - 40 = 10$$
$$I_1R_1 = 50 \times 0.2 = 10$$
$$I_0R_0 = 10 \times 0.2 = 2$$
$$I_2R_2 = 40 \times 0.2 = 8$$
$$V_A = E_1 - I_1R_1 - I_0R_0 = 115 - 10 - 2 = 103 \text{ volts}$$
$$V_B = E_2 + I_0R_0 - I_2R_2 = 115 + 2 - 8 = 109 \text{ volts}$$

The load voltages V_A and V_B are unbalanced as is to be expected with $I_1 \neq I_2$.

Now let us consider I_2 reduced to 10 amp. Then

$$I_0 = I_1 - I_2 = 50 - 10 = 40$$
$$I_1R_1 = 50 \times 0.2 = 10$$
$$I_0R_0 = 40 \times 0.2 = 8$$
$$I_2R_2 = 10 \times 0.2 = 2$$
$$V_A = 115 - 10 - 8 = 97 \text{ volts}$$
$$V_B = 115 + 8 - 2 = 121 \text{ volts}$$

We now find that not only is $V_B > V_A$ as before but also $V_B > E_2$ and customer B may well be annoyed with excessive lamp replacements. It is clear from the computation that this situation occurs when $I_0 > I_2$ because, for load B, I_0R_0 is then a potential *rise* instead of the potential *drop* too commonly taken to be inviolate for IR potentials.

Any considerable reduction in wire size for the neutral on the premise that it is unlikely to carry much current may prove unwise. It is the practise to provide for sustained and complete unbalance by making the neutral wire the same size as the other two.

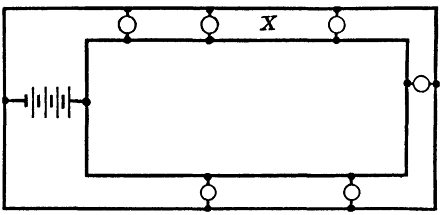

FIG. 8·2. A loop distribution circuit.

8·6. The Loop System. Continuity of service is often a major consideration and the loop circuit promotes this objective in addition to combating voltage drop. Any one break in the line as at X, Fig. 8·2, will affect only the voltage drop without interrupting service.

This circuit is commonly used for distribution of power *in* congested city areas as well as for transmission of power *to* cities. Ithaca is fed

from a loop which includes generating stations in Elmira and Binghamton. This, however, is not an *isolated* loop since it is connected through to Niagara and to New York City as well, thus comprising a more formidable circuit composition commonly known as a *network*.

8·7. Other Systems. The radial system, as the name suggests, feeds in all directions from a *load center* which is judiciously located to provide a minimum of kilowatt-miles. It is now largely supplanted by the network or multiloop system which today is by far the most common large-scale system. Treatment of networks will be reserved for a later study.

8·8. Voltage Regulation. Voltage regulation might seem merely to be an expression referring to the control of voltage for constancy. The term, however, has a very definite connotation other than the obvious and is commonly expressed in per cent. ***Per cent voltage regulation is the per cent rise in voltage occurring at a load as the load is removed,*** the source (generator) voltage remaining constant. For a simple line with a single concentrated load it is

$$\%\ \text{V.R.} = \frac{E_g - E_L}{E_L}$$

where E_g and E_L are the generator and load voltages, respectively, when the line is loaded. It is an algebraic quantity (note *plus* for *rise* in voltage) and for a-c lines may actually be negative.

8·9. Efficiency. The power efficiency of a transmission line, like the efficiency of other apparatus, is the *ratio* of the power *output* to the power *input* of the line. Care is at times necessary to avoid tempting short cuts too early in the computation.

Confusion sometimes arises when it is not kept firmly in mind that the "output" and "input" for an efficiency computation refer strictly to the particular equipment for which the efficiency is desired. ***The input*** *to the line* is often the *output of a generator* and ***the output*** *of the line* is likely the *input to a load.* Careless analysis may thus lead to an inverted result but this should be apprehended if it is realized that power efficiency can neither exceed nor attain 100 per cent. The power loss in the line, of course, is best computed as I^2R where possible. When this and the load power are known,

$$\text{Efficiency } \eta = \frac{P_{\text{load}}}{P_{\text{load}} + \Sigma I^2 R_{\text{line}}}$$

8·10. Economic Requirements. The economic size of wire, like many other economic problems, involves two basic cost items:

1. Operating cost—all items which depend on (vary with) the amount of energy conducted.

2. Fixed cost or "overhead"—all items which do not depend on the amount of energy conducted.

When these items are governed by the following assumptions, a simple relation known as ***Kelvin's law for most economic conductor size*** can be formulated.

1. The annual operating cost is inversely proportional to the conductor cross section $\$_{op} = k_1/A$. This assumes that the operating cost is principally the cost of the energy lost in heating the conductors.

2. The annual fixed cost is directly proportional to the conductor cross section, $\$_f = k_2A$. This assumes that the annual fixed cost is principally the *annual* interest, depreciation, insurance, etc., on the investment in conductor metal, which may be expressed as a fixed per cent of the investment (annual installment).

The total annual cost is $\$_t = \$_{op} + \$_f = \dfrac{k_1}{A} + k_2A$.

Differentiating to find the conductor cross section A for minimum $\$_t$,

$$\frac{d}{dA}\,\$_t = -\frac{k_1}{A^2} + k_2 = 0$$

This may be arranged to give

$$\frac{k_1}{A} = k_2A$$

which reduces simply to

$$\$_{op} = \$_f$$

As seen in Fig. 8·3 this means that minimum $\$_t$ occurs where $\$_f$ and $\$_{op}$ intersect. Thus Kelvin's law states that

The most economic conductor cross section is that one which makes the operating and fixed costs equal.

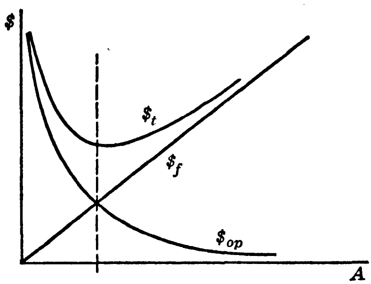

FIG. 8·3. Graph of costs for Kelvin's law.

Note carefully that the conditions upon which the law is formulated are highly *idealistic* and that the actual operating and overhead costs may depart sufficiently from the simple functions assumed in the law to make it inapplicable. It should be observed further that Fig. 8·3 does not admit of indiscriminate generalizing; in general the sum of two intersecting curves is a minimum directly over their point of intersection *only* when at this point the negative slope of one curve numerically equals the positive slope of the other.

The economic problem is by no means confined to wire size, important as that may be. For either interior or exterior lines, the choice of the material and fabrication of the conductor, of its insulation, design of supporting structure, protection equipment, etc., together with the weighing of costs of equipment against costs of installation; installed cost against cost of maintenance, cost of energy loss and value of reliability is no mere Kelvin's law problem but is a study in itself.

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QUESTIONS

8·1. (*a*) What are the two basic circuits used for electric distribution?

(*b*) What is the outstanding disadvantage of each?

8·2. What four basic factors should in general be considered in selecting the proper wire size for conveyance of power?

8·3. What is the National Electrical Code and how does it influence the selection of wire for interior distribution?

8·4. Derive and *state* Kelvin's law. Expound its *limitations.*

8·5. What serious error is most likely to be committed in planning the wiring of a new building?

8·6. Describe the three-wire Edison system and its purpose.

8·7. Why is the neutral wire of a three-wire Edison system not fused?

8·8. Explain what conditions will cause a load voltage in a three-wire Edison system to exceed the source (or generator) voltage.

8·9. A two-wire line of resistance R supplies a single load with power P at voltage V. Find line efficiency $\eta = f(V,P,R)$ and show that with P and R fixed, efficiency η improves as V is increased.

8·10. For a given load power and line efficiency derive a relation to show how the size of wire required for a two-wire line depends upon the load voltage used.

8·11. Define voltage regulation.

8·12. Account for the *generator* voltage E_g appearing in the expression given for computing the per cent voltage regulation at the *load.*

8·13. What is the meaning of a negative voltage regulation?

8·14. Given a two-wire line with sending-end voltage E_1 and resistance R_L *fixed,* determine:

(*a*) The maximum power (in terms of E_1 and R_L) that can be *delivered* to the load.

(*b*) The line efficiency.

(*c*) The voltage regulation.

PROBLEMS

8·1. Prove that, according to Kelvin's law, minimum total annual cost is obtained when the size of conductor A_0 in circular mils, for a given current I, in use

H hours per year, is

$$A_0 = K \times I \left[\frac{C_e \times H}{C_c \times F}\right]^{1/2}$$

where C_e = cost of energy wasted in the line, in cents per kilowatthour,
C_c = cost of copper installed, in cents per pound of bare wire,
F = a fraction representing the rate for fixed charges.

Show that, for annealed copper, the value of $K = 59$; and that if aluminum is used, $K = 137$.

8·2. A feeder supplies a motor 200 ft distant from a panel at which the voltage is maintained at 115 volts. If the motor requires a current of 60 amp and the voltage drop is limited to 5 per cent of the supply voltage, what is the minimum size (in circular mils) of copper wire that can be used? (Temperature, 25° C.)

(*a*) If the initial cost of installation (including copper wire, insulation, labor, and supplies) is 90 cents per pound of bare wire, the cost of energy 3 cents per kilowatthour, and the fixed charges 12 per cent of the initial cost, what is the most economical size wire to use? The motor operates at full load continuously.

(*b*) What is the smallest size of wire allowed on the basis of safe current-carrying capacity?

8·3. Compare the power saving in line I^2R loss achieved by using a three-wire Edison line (wires equal size) instead of a plain two-wire line, the lines to have the same length, to use the same total amount of copper, and to supply equal power P at the same voltage V. The power from the three-wire line is to be applied to two loads P_a and P_b, each at voltage V, the one from wire a to neutral, the other from wire b to neutral. The voltages V are assumed to be maintained constant at the loads by such adjustment of the sending-end voltages as changes in the loads require.

The saving achieved by the three-wire line depends on the degree of load unbalance. Let the unbalance be expressed as a ratio $m = P_a/P_b$. Sketch the circuit for each line and determine the following:

(*a*) For one wire of each line the ratio of wire section A_3/A_2 (three-wire to two-wire) and of wire resistance R_3/R_2.

(*b*) The ratio x of the total I^2R line loss in the three-wire line to that in the two-wire line, as a function of m.

(*c*) Find the per cent $y = f(m)$ of the two-wire line I^2R loss which is saved by substituting the three-wire line.

(*d*) Sketch the graph for $y = f(m)$ found in (*c*). Evaluate in particular the points: $m = 0$, $m = 1$, $m = -1$, $y = 0$, and $y =$ maximum.

(*e*) Explain the physical significance of the first three points given in (*d*) and justify the values obtained for y in each.

(*f*) Because of the interchangeability of the unbalance in the symmetrical three-wire line, the same relation should be obtained for $y' = f(n)$ as for $y = f(m)$ where $n = 1/m$. Substitute $n = 1/m$ in $y = f(m)$ and check. Check also the two values of m found in (*d*) for $y = 0$ and show that they are reciprocal quantities.

8·4. To supply the d-c power requirements in a long factory building a 550-volt, two-wire line is being planned. The *line* will be 1000 ft long (each wire) and the load will be fairly uniformly distributed along it at the *rate* of approximately 0.2 amp per foot of *line*. For economy of copper it is planned to reduce the wire size in steps of 250 ft, beginning at the supply panel, so that uniform current density will be approximated all along the line. No voltage less than 525 volts at any load and only the amount of copper required to meet this minimum is to be permitted.

(*a*) As a preliminary computation, obtain the mathematical solution for an equivalent *idealized* line, i.e., assuming its cross section reduced *uniformly* instead of in steps. Find the circular mil size of each end of the idealized line.

(*b*) Translate the findings in (*a*) to *average* conductor area for each 250-ft section of the line and select the nearest commercial wire sizes. *Note:* From No. 0 to No. 20 only the *even* B&S sizes are regularly stocked.

(*c*) Compute the actual *IR* drop in each section of line found in (*b*) and check the total *IR* drop against the 25 volts allowed.

(*d*) What effect, if any, will the NEC requirements have on the proposed installation?

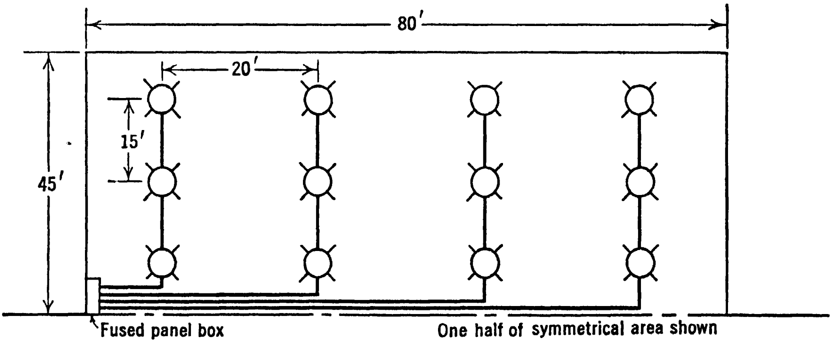
Note: It may be of interest that while minimum *power* loss is obtained, as here, with *uniform* current density, minimum *voltage* drop is obtained with current density *proportional* to sectional area of wire. Stated in other terms, it can be shown that the quantity to be minimized must be proportional to the square root of the distance from the far end (zero current end) of the line. For the ideal smooth taper, similar to part (*a*), the voltage drop, treated as a minimum, can be kept down to the required 25 volts with $\frac{8}{9}$ of the copper required for uniform current density.

8·5. A conductor of material with $\rho = 50$ ohms per circular mil foot is tapered at the uniform rate of 10 circular mils per foot. What will be the resistance of a length of the conductor which at one end has double the cross section of the other end?

8·6. A distribution system for lighting an industrial area is desired as shown in plan view below. Each outlet is to supply 750 watts with 2 per cent maximum permissible drop from the 115-volt panel box.

(*a*) Determine the wire size required for each run from panel to outlet, and outlet to outlet, labeling each with the B&S gage number.

(*b*) Select the proper sizes of conduit and indicate them on the diagram.



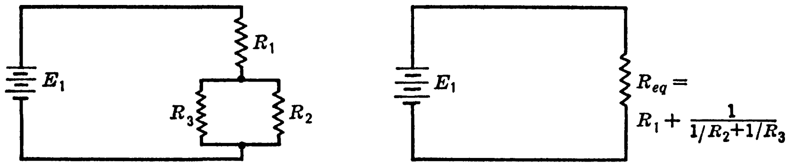
(*c*) If the panel box is fed by a three-wire 230/115-volt feeder from a distance of 50 ft, what size B&S wire and conduit must be used to keep within 1 per cent voltage drop?

CHAPTER IX

ELECTRICAL NETWORKS

The circuits employed in power, communications, and in most electrical work today are commonly of higher complexity than those discussed in the previous chapter. The generic term for the several classifications of these is *networks*. We now proceed to develop some of the tools commonly used for attacking the problems which such circuits present.

9·1. Limitations of Equivalent Resistance. When two or more resistances are connected in series or in parallel or in series-parallel as in



(a) (b)

FIG. 9·1.

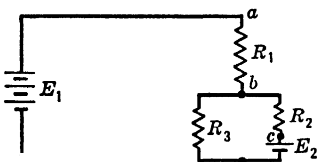


FIG. 9·2.

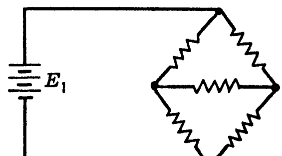


FIG. 9·3.

Fig. 9·1 the computation of unknowns in the circuit is commonly simplified by substituting an equivalent resistance for each such group of resistances. It is elementary physics that "resistances in series add directly" and "resistances in parallel add as reciprocals" so that the equivalent resistance for Fig. 9·1*a* is as shown in Fig. 9·1*b*. Although these are unquestionably very useful facts it is important to appreciate that they are of *limited* application and *to understand when they are inapplicable.*

In Fig. 9·2, for example, the introduction of E_2 definitely destroys the simple parallel connection of R_2 and R_3 so that there is *no equivalent*

resistance that can be substituted for them and the method is inapplicable (except by the principle of superposition to be later discussed). We thus observe that the equivalent resistance attack fails when any parallel component of the resistance combination includes a source of emf.

In Fig. 9·3 is shown a common connection of resistances which cannot be described as a series-parallel combination. Although it is possible to compute an equivalent resistance for this case, a more powerful attack than the simple summation of series and parallel equivalents is required. This also will be developed presently.

Without laboring the point further it is evident that there is need to re-examine the basic principles involved in the specialized series-parallel cases and to produce more fundamental and more widely applicable methods of attack. Let us first consider the most important of these which is formulated in the well-known *Kirchhoff's laws* (note the two h's).

9·2. Kirchhoff's Laws. It may be recalled that the rules for the series and parallel combinations above mentioned are based upon two very fundamental concepts as follows.

1. Electricity behaves as though incompressible; the electron displacement which constitutes current in a conductor is inelastic and neither adds to nor subtracts from the number of electrons anywhere within the conductor.

2. The difference in potential between any two points in (or outside of) an electric circuit is independent of the path between the points.

The statement of the first law is focused upon those parts of an electric circuit where three or more conductors are joined together in what is termed a *junction* as at b or d in Fig. 9·2 and is as follows.

a. ***The algebraic sum of all currents flowing into (or out of) a junction is zero. Symbolically,*** $\Sigma I = 0$.

The second law is commonly expressed in terms of a loop or closed traverse which may be seen to embody conveniently the principle in (2). Consider any two points on the loop such as b and d of loop $bcdb$ in Fig. 9·2.

Since the same potential difference is found between the two points by any path between them, it follows that a round trip inventory of potential differences via $bcdb$ will algebraically total zero. It further follows that the introduction of any number of intermediate points such as c in the loop will not alter the total inventory of zero. The rise and fall of potential encountered in the traverse may be likened to the rise and fall of elevation encountered in the *traverse* of a closed line in *surveying* where it is quite clear that the algebraic summation of all the changes in elevation encountered in the closed traverse should be precisely zero. The statement of the second law is then:

b. The algebraic sum of all potential differences taken successively around a closed traverse is zero. Symbolically $\Sigma pd = 0$ *or* $\Sigma V = 0$.

It is well to observe that the above statement avoids mention of a closed circuit. The electric circuit *need not be closed* and there need be no current in order to make this law either valid or useful. Referring again to the surveying analog, it is necessary only that the *traverse* be closed; not that the surveyor follow a traffic loop or even complete his circuit by an improved trail or path of least resistance. Consider the circuits of Fig. 9·4.

In Fig. 9·4*a* while switch *s* is open the circuit is not closed and there is no current. It is entirely possible that we may wish to know the

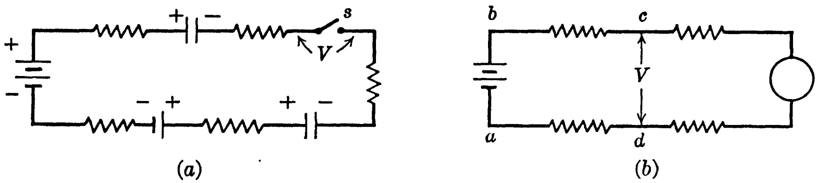


FIG. 9·4.

potential difference V across the switch s. The application of Kirchhoff's $\Sigma V = 0$ is not only valid but also is difficult to improve upon as a sure-fire attack.

Figure 9·4*b* shows a simplified case where it is desired to find the potential difference V for which two adjacent points c and d of a closed two-wire line must be insulated. A traverse around *abcda* is the rational application of Kirchhoff's $\Sigma V = 0$ although again the traverse is not confined to the closed electric circuit.

The loop and junction equations obtained by applying Kirchhoff's laws commonly comprise all or part of a set of independent simultaneous equations which promote the solution of various electric circuit problems. It is important that the *objective* in any particular problem be kept clearly in mind during the application of Kirchhoff's laws and throughout the subsequent mathematical processing. Failure to do this commonly results in getting nowhere by going in figurative circles as though our surveyor had become lost.

Number of Kirchhoff Equations. It is by no means desirable to write all possible loop and junction equations, because only a definite and easily determined number of these equations can be independent and can definitely contribute to the solution. It can be shown that *the number of junction equations should be* **one less** *than the total number of* **junctions** and that *the number of loop equations should be* **equal** *to the*

*number of **meshes*** or undivided enclosures bounded by the conductors in the drawing of the circuit.

Writing the Loop Equations. Pursuing the surveying analog it is common knowledge that the rises and falls of elevation are taken in their natural sequence around the loop. We would indeed rate him low who proceeded to measure first all the rises in elevation around the traverse and then repeat the traverse to pick up all the drops in elevation. Clearly the *orderly pursuance of the traverse* from one element to the next until it is closed will provide minimum opportunity for error.

In the electrical case, however, it is not at all uncommon to find our loop equation expressed as $\Sigma E = \Sigma IR$, the sum of the emf rises equals the sum of the IR drops. True as the statement may be it does not follow that it is good engineering procedure to pluck first all the E's and then all the IR's. In the first place, the direction in which we make the traverse affects all algebraic signs so that any second traverse of the same loop must be in the same direction. Secondly, when the potentials are not picked up in succession, it is possible to miss one or more unless the circuit is extremely simple. For these reasons it is simply good engineering to write $\Sigma V = 0$ and to pick up these V's in succession around the traverse.

It is important to *identify* all voltages clearly on the circuit diagram as in Fig. 9·5*b*. Until experienced it is well to show the known (or assumed) *polarities* of each voltage, bearing in mind that the polarity must be consistent with the nature of the circuit element concerned. Through a resistance or other load element conventional current is always directed from + to −; through a source element it is always directed from − to +. Algebra cannot correct for all wrong assumptions because it is sometimes powerless to produce the necessary minus sign when it is needed.

Although it is true that the requirements of the particular problem may indicate that certain itineraries of traverse are best, it is *generally* helpful, when the circuit is well drawn, to *follow the boundary of each mesh* rather than to explore as much of the circuit as possible in any one traverse. This tends to keep down the number of terms in each equation and to simplify their subsequent manipulation.

Summary of Procedure. The following summary of a general procedure may not always lead to the shortest solution but in lieu of practice it is likely to provide the quickest route to a correct solution.

1. Make a circuit diagram.
2. Label diagram fully.

a. Show by arrows all given current directions and identify each with subscripted I.
b. Show by + and − all given voltage polarities and identify each with subscripted E or V.

3. *Assume* and *identify* as in (2) direction of each *unknown* current and polarity of each *unknown* voltage.
4. *Agree* that algebraic plus denotes:
   a. Current *to* a junction.
   b. Potential *rise.*
5. Write Kirchhoff equations for:
   a. All junctions but one.
   b. All meshes.
6. Write all pertinent relations between voltages and currents, applying Ohm's law, $P = EI$, etc., as may be required.
7. Having obtained as many equations as unknowns, solve with discretion as to easiest algebraic process.

Note particularly that the *circuit sketch is required.* To write equations for a circuit without direct identification of them with the sketch is an abuse of the imagination not to be tolerated here.

Example. Consider the circuit of Fig. 9·2 with known quantities as labeled in Fig. 9·5*a*, to find all currents.

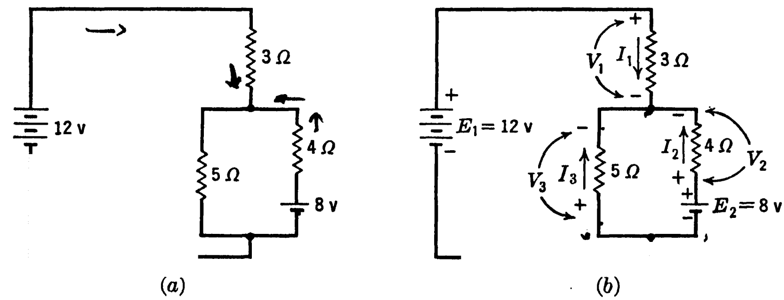


FIG. 9·5.

Figure 9·5*b* shows one version of executing the first three items of the procedure just outlined. Items 4 and 5 follow.

(*a*) $+I_1 + I_2 + I_3 = 0$ (junction *b*)

(*b*) $+E_1 - V_1 + V_3 = 0 \qquad 12 - 3I_1 + 5I_3 = 0$

(*c*) $-V_3 + V_2 - E_2 = 0 \qquad -5I_3 + 4I_2 - 8 = 0$

For solution it is discreet to observe that substitution of (*b*) and (*c*) in (*a*) will eliminate all but I_3, as follows.

$$I_1 = \frac{12 + 5I_3}{3}, \qquad I_2 = \frac{8 + 5I_3}{4}$$

$$\frac{12 + 5I_3}{3} + I_3 + \frac{8 + 5I_3}{4} = 0 \quad \text{or} \quad I_3 = -1.532 \text{ amp}$$

Then

$$I_1 = \frac{12 - 7.66}{3} = 1.447 \quad \text{and} \quad I_2 = \frac{8 - 7.66}{4} = 0.085 \text{ amp}$$

Checking in (*a*), $1.447 + (-1.532) + 0.085 = 0$

This, of course, is only one of a diversity of problems too great to permit classification of them into *types* that might expedite their solution. The application of Kirchhoff's laws, and the solution of circuit problems in general, admit of no substitute for *basic analytical methods;* practice and experience, of course, are required for rapid and accurate work.

9·3. Use of Determinants. The set of simultaneous equations obtained from Kirchhoff's and other laws of electric circuits is often best solved by determinants. Although the circuit example just considered is unduly simple for the purpose it may be so solved as follows.

Given:

$$\begin{aligned} I_1 + I_2 + I_3 &= 0 \\ 3I_1 + 0 - 5I_3 &= 12 \\ 0 + 4I_2 - 5I_3 &= 8 \end{aligned}$$

$$I_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 12 & 0 & -5 \\ 8 & 4 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -5 \\ 0 & 4 & -5 \end{vmatrix}} = \frac{-12\begin{vmatrix} 1 & 1 \\ 4 & -5 \end{vmatrix} + 8\begin{vmatrix} 1 & 1 \\ 0 & -5 \end{vmatrix}}{1\begin{vmatrix} 0 & -5 \\ 4 & -5 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 4 & -5 \end{vmatrix}}$$

$$= \frac{-12(-5 - 4) + 8(-5)}{1(+20) - 3(-5 - 4)} = \frac{+108 - 40}{+20 + 27} = \frac{68}{47} = 1.447 \text{ amp}$$

$$I_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 3 & 12 & -5 \\ 0 & 8 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -5 \\ 0 & 4 & -5 \end{vmatrix}} = \frac{1\begin{vmatrix} 12 & -5 \\ 8 & -5 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 8 & -5 \end{vmatrix}}{47}$$

$$= \frac{1(-60 + 40) - 3(-8)}{47} = \frac{4}{47} = 0.085 \text{ amp}$$

$$I_3 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 12 \\ 0 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -5 \\ 0 & 4 & -5 \end{vmatrix}} = \frac{1\begin{vmatrix} 0 & 12 \\ 4 & 8 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 4 & 8 \end{vmatrix}}{47}$$

$$= \frac{-48 - 24}{47} = \frac{-72}{47} = -1.532 \text{ amp}$$

9·4. Other Methods. While it should be kept firmly in mind that Kirchhoff's laws formulate the *basic* means of attack for electrical networks in general, there are several alternative methods applicable to numerous *special cases* which so shorten the labor of solution that mastery of them is well repaid. All these in some manner are concerned with the principle of superposition. While this principle is common knowledge, it is so frequently misapplied by reason of superficial analysis that it seems advisable to deal with the matter more carefully than its apparent simplicity might otherwise warrant.

9·5. The Principle of Superposition. When a variable quantity $y = f(x)$ is a linear function of x we write $y = a + kx$ and, in rectangular coordinates, graph a straight line of slope k through point

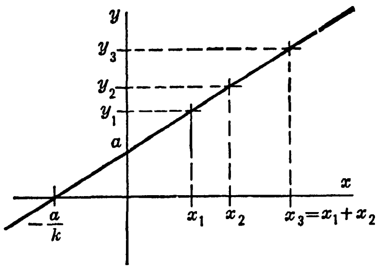


FIG. 9·6.

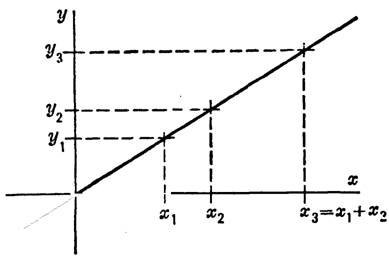


FIG. 9·7.

$y = a$ (Fig. 9·6). If $a = 0$, then $y = kx$, and the straight line goes through the origin (Fig. 9·7). We then say that y is directly proportional to x.

Direct proportionalities have many unique and useful properties. The student is introduced early to some of these in his study of *proportion* in elementary school arithmetic. The familiar relation is easily developed from $y = kx$ by equating quotients $k = y_2/x_2 = y_1/x_1$ to obtain "y_2 is to x_2 as y_1 is to x_1" where y_2, x_2, y_1, and x_1 are as shown in Fig. 9·7. There seems to be a tendency to ignore the fact that $y = kx$ provides the proportion $y_2/x_2 = y_1/x_1$ or $y_2/y_1 = x_2/x_1$ and to

compute the value of k as a means of finding new values of $y = f(x)$ from a known value. *This tendency is to be deplored* not only because computation of the intermediary k increases the chance of error but also because any concept of the behavior or dynamics of the phenomena is thereby lost. It is quite as much our purpose to *develop these concepts* as it is to attain proficiency in the computation of specific cases.

The particular aspect of proportionality which is of immediate concern in *superposition* is based upon the *summation* rather than the quotient or ratio of proportional quantities. Webster defines superposition as the act of "superimposing" or "placing upon" in a vertical tier; perhaps *to stack* as in the laying up of bricks provides the clearest picture.

Returning to $y = kx$ and Fig. 9·7 we again consider the particular values $y_1 = kx_1$ and $y_2 = kx_2$. Let us be clear that these mean that the x_1 value of x gives value y_1 for y while x_2 value of x gives value y_2 for y.

Now if we superimpose value x_2 on x_1 so that x becomes $x_1 + x_2 = x_3$ what value y_3 will y have in terms of y_1 and y_2? The answer is disarmingly simple because we readily obtain

$$y_3 = kx_3$$

$$y_3 = k(x_1 + x_2) = kx_1 + kx_2$$

$$y_3 = y_1 + y_2$$

In terms of cause and effect this means that when effect y is directly proportional to cause x, the sum x_3 of causes x_1 and x_2 produces effect y_3 which is simply the sum of the effects y_1 and y_2 produced by the respective causes x_1 and x_2. There is nothing new or startling in this statement and it may easily escape the attention which it really deserves. Let us turn therefore to a physical illustration.

We shall put x measured gallons of water into a pail or can which has a flat horizontal bottom and vertical sides, and measure the depth y of water. By reason of the shape of the pail the depth of water (effect) is directly proportional to the quantity of water (cause), or $y = kx$. If x_1 gallons produce depth y_1, and x_2 gallons produce depth y_2, the superposition of x_2 gallons on x_1 gallons in the pail gives depth $y_1 + y_2$. Again the result is not exciting because we can so easily predict the depth which any quantity of water in the pail will produce.

Now let us observe the consequence of distorting the pail as by a dent either in bottom or side. Again we put in x_1 gallons and measure depth y_1. Then we substitute x_2 gallons and measure y_2. If we put $x_1 + x_2$ gallons in the pail what now will be the depth? The possibility that the depth may be $y_1 + y_2$ is indeed too remote to entertain! Even

if the pail is a perfect one, but with conventionally sloping sides, the principle of superposition is inapplicable and $y_3 \neq y_1 + y_2$. Considering the variety of shapes of our commonly used containers it might startle one a bit to face the fact that formulation of a mathematical relation between quantity and depth could be really embarrassing. In such a dilemma the simplicity of superposition or proportionality assumes its true stature and importance; we appreciate that it is really something quite special and choice, not to be blandly taken for granted.

For final emphasis on the *uniqueness* of the direct proportionality and the dependent principle of superposition we shall briefly survey two simple mathematical functions which depart from it. First we return to Fig. 9·6 which, although linear, is not a direct proportionality. Again we set up the particular values $y_1 = a + kx_1$ and $y_2 = a + kx_2$, let $x_3 = x_1 + x_2$ and find

$$y_3 = a + kx_3$$

$$y_3 = a + k(x_1 + x_2) = a + kx_1 + kx_2$$

$$y_3 = (a + kx_1) + (a + kx_2) - a$$

$$y_3 = y_1 + y_2 - a$$

$$y_3 \neq y_1 + y_2$$

Thus we observe that *linearity is not sufficient* to insure the direct applicability of superposition; the axes must be so chosen as to make $a = 0$ and provide a *direct* proportion.

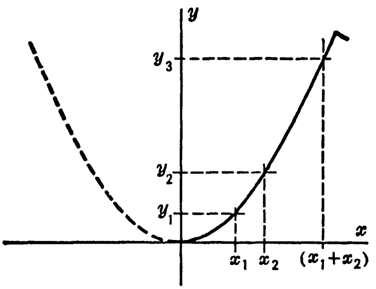


Fig. 9·8.

Lastly let us consider the parabolic relation of Fig. 9·8 which is expressed by $y = kx^2$.

If we again let $x_3 = x_1 + x_2$ we find

$$y_3 = kx_3^2$$

$$y_3 = k(x_1 + x_2)^2 = kx_1^2 + kx_2^2 + 2kx_1x_2$$

$$y_3 = y_1 + y_2 + 2kx_1x_2$$

$$y_3 \neq y_1 + y_2$$

While the foregoing is no actual proof, we are reasonably prepared to acknowledge the demonstrated limitation of the principle of superposition to cases of direct proportionality and to reassert that: ***The principle of superposition states that when cause and effect are directly proportional, the effect of a composite cause is equal to the sum of the effects of the component causes.***

Application to Electric Circuits. Because Ohm's law, $E = RI$, is a direct proportionality between E and I, like $y = kx$, it is clear that electrical networks which comprise only sources of emf and constant resistances are amenable to applications of the principle of superposition. We shall consider four methods which are concerned with this concept.

1. Maxwell's mesh method.
2. The method of superposition.
3. Thevenin's theorem.
4. The Y-delta transformation.

9·6. The Maxwell Mesh Method. The Maxwell mesh method is a simplified procedure for applying Kirchhoff's laws to networks comprising

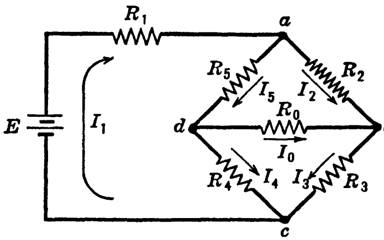


FIG. 9·9.

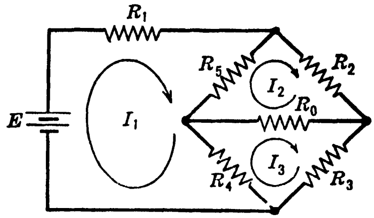


FIG. 9·10.

only emf's and resistances which are independent of the respective currents through them and which are therefore subject to the principle of superposition. We shall develop the method by applying Kirchhoff's laws to the specific case of Fig. 9·9. For each branch current a direction is chosen as shown and the junction equations are

For *a*

$$I_1 - I_2 - I_5 = 0 \qquad [9\cdot1]$$

For *b*

$$I_2 - I_3 + I_0 = 0 \qquad [9\cdot2]$$

For *c*

$$I_3 + I_4 - I_1 = 0 \qquad [9\cdot3]$$

The loop equations are

For *cadc*

$$E - R_1I_1 - R_5I_5 - R_4I_4 = 0 \qquad [9\cdot4]$$

For *cdbc*

$$R_4I_4 - R_0I_0 - R_3I_3 = 0 \qquad [9\cdot5]$$

For *dabd*

$$R_5I_5 - R_2I_2 + R_0I_0 = 0 \qquad [9\cdot6]$$

Now let us rewrite the junction equations (9·1–9·3) to give

$$I_5 = I_1 - I_2 \qquad [9\cdot 7]$$

$$I_0 = I_3 - I_2 \qquad [9\cdot 8]$$

$$I_4 = I_1 - I_3 \qquad [9\cdot 9]$$

Substituting these in the loop equations (9·4–9·6) gives

$$E - R_1I_1 - R_5(I_1 - I_2) - R_4(I_1 - I_3) = 0 \qquad [9\cdot 10]$$

$$R_4(I_1 - I_3) - R_0(I_3 - I_2) - R_3I_3 = 0 \qquad [9\cdot 11]$$

$$R_5(I_1 - I_2) - R_2I_2 + R_0(I_3 - I_2) = 0 \qquad [9\cdot 12]$$

Collecting and transposing terms in each equation gives

$$(R_1 + R_5 + R_4)I_1 - R_5I_2 - R_4I_3 = E \qquad [9\cdot 13]$$

$$-R_4I_1 - R_0I_2 + (R_4 + R_0 + R_3)I_3 = 0 \qquad [9\cdot 14]$$

$$-R_5I_1 + (R_5 + R_2 + R_0)I_2 - R_0I_3 = 0 \qquad [9\cdot 15]$$

The three loop equations are thus worked into shape for forming determinants for the solution of I_1, I_2, and I_3 in terms of E and the resistances.

The foregoing is nothing more than a normal process for the application of Kirchhoff's laws. We are now in position, however, to observe the nature of Maxwell's contribution to the procedure. Instead of putting branch currents such as I_5, I_0, and I_4 on the circuit diagram, Maxwell conceived the use of *mesh* currents, such as I_1, I_2, and I_3 of Fig. 9·10, each *continuous throughout the closed traverse of its mesh boundary*. So long as R_5, R_4, and R_0 are each independent of the current through them, this concept is but a new interpretation of the meaning of equations 9·7 to 9·9. Equation 9·7, for example, now signifies the superposition of current components I_1 and I_2 which flow through R_5 as an equivalent of I_5, and is no longer construed to be a junction equation.

Although it is true that this concept reduces the determinant from the sixth to the third order, this was also accomplished by appropriate manipulation of the Kirchhoff equations and is not a unique contribution of the Maxwell mesh method.

To discover and appreciate the distinctive advantage provided by the method we must turn to equations 9·13–9·15 and observe that they follow a significant *pattern* with respect to the mesh components of the circuit in Fig. 9·10 such that by following this pattern the ***final equations can be written directly from the circuit diagram*** without the

preliminary labor of the preceding twelve equations (9·1–9·12) entailed in the normal application of Kirchhoff's laws.

Equation 9·13 concerns the mesh traversed by I_1 which we shall call mesh 1. The first term is the sum of all IR drops in mesh 1 due to I_1. It is simply I_1 times the arithmetic sum of the resistances traversed by I_1. The second term is the sum of all IR drops in mesh 1 due to I_2. It is minus because I_2 has direction opposite to I_1 through the mutual resistance R_5. The third term, like the second, is the sum of all IR drops in mesh 1 due to I_3 and is minus because I_3 has direction opposite to I_1 through the mutual resistance R_4. The last term is the sum of all emf rises traversed by I_1.

Equations 9·14 and 9·15 may be seen to follow the same pattern for meshes 3 and 2 respectively.

If all the mesh currents are taken to have the ***same direction of traverse*** (clockwise in this instance) determination of the algebraic signs for the IR's is simplified to the mere routine of plus for terms containing what we may call the "home" current (I_1 in equation 9·13, I_3 in equation 9·14, etc.) and minus for all other terms.

Thus it may be seen that the Maxwell mesh method, when applicable, substitutes for the conventional Kirchhoff law procedure a simple routine for writing directly a set of equations, not only of minimum number, but also in such form as enables solution by determinants with no algebraic manipulation whatsoever.

Summary of Procedure

1. Make a circuit diagram and put all known values on it.
2. For each mesh in the network assume a mesh current, show by arrow its assumed direction, and label it I_1, I_2, I_3, etc. Equation writing is facilitated by assuming the mesh currents either all clockwise or all counterclockwise, as noted below.
3. For each mesh write a voltage equation comprising on one side a sum of IR terms, one for each mesh current in the network, and on the other side the algebraic sum of all the emf rises traversed by the mesh current for that mesh. The IR terms consist of the following.

   a. The mesh current of the mesh in question times the arithmetic sum of all the resistances which it traverses.

   b. For each other mesh current its product with the arithmetic sum of all the resistances which it traverses in *common* with the mesh current of (*a*). If all mesh currents are taken in the same direction, as suggested in 2, the IR term from (*a*) may be taken plus and all the IR terms from (*b*) will then be minus.

4. If the IR terms of each equation in 3 are arranged in the same sequence, according to their mesh current subscripts, they readily provide the determinants required to solve for the mesh currents.

Application. The Maxwell mesh method is widely used. When the complete solution of a network is desired and, as commonly occurs, the restriction, $R \neq f(I)$, is not violated, this method is not only applicable but also it is likely to afford the best means of solution.

9·7. The Method of Superposition. When a network consists only of emf sources and constant resistances, the current in any part can be found by algebraic summation of the current which would be produced there by each emf separately, zero resistance being substituted for each emf removed. (It is assumed here that any internal resistance of the sources is included in the resistances of the network.)

This *application of the principle* of superposition is known as the *method* of superposition. If an equivalent resistance can be found for the application of each emf, the computation may not even require solution of simultaneous equations. Let us again utilize the circuit of Fig. 9·2 for illustration of the method, and redraw it as shown in Fig. 9·11*a*.

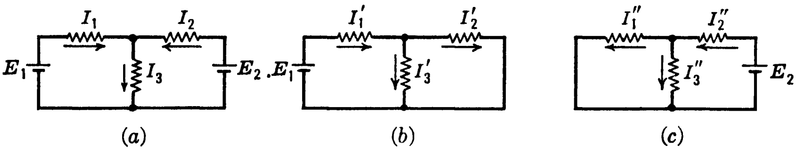


FIG. 9·11.

The application of E_1 and E_2 respectively are shown in Fig. 9·11*b* and 9·11*c* with assumed direction of each current shown. The solution of these circuits for the several currents is elementary. We then find for the original circuit

$$I_1 = I_1' - I_1'' \quad [9\cdot16]$$

$$I_2 = I_2'' - I_2' \quad [9\cdot17]$$

$$I_3 = I_3' + I_3'' \quad [9\cdot18]$$

Application. For the complete solution of a network the method of superposition is likely to prove more laborious than the Maxwell mesh method. Furthermore, precision suffers by comparison when the resultant actual current happens to be much smaller than either of its components. It is advantageous when we wish to determine the consequence of modifying a given network by the addition (insertion) of an emf in some branch.

As a simple example of the latter let us assume that Fig. 9·11*b* represents the given network which is to be modified by the addition of an emf E_2 as in Fig. 9·11*a*. By solving Fig. 9·11*c* the currents I_1'', etc., thus found represent the *change* in currents (and *IR*'s) which will result in the respective branches of the circuit. When these changes in current are relatively *small*, the method is likely to afford higher precision results than other methods, for computation with a given size of slide rule or other computing aid.

It should be clear that, as a general means for solving network problems which conform to superposition restrictions, this method may be at some disadvantage when the currents are among the given quantities. To predict whether the method may be advantageous or not evidently requires more than superficial judgment.

9·8. Thevenin's Theorem. Let us assume that we have a network of any degree of complexity, but *restricted exclusively to constant emf's and constant resistances*, enclosed in a box so that only *two terminals* to the network are accessible. In general there will be a voltage between these terminals, and it will be a function of the current between terminals, which is subject to control from elements *outside* of the box. Because the network is restricted to constant emf's and resistances, the principle of superposition and the method of superposition apply. However complex the contents of the box may be, it can be resolved by the method of superposition into as many simple components of one emf applied to an equivalent resistance as there are emf's in the box. Since *each* of these components has a linear voltage-current characteristic, it becomes impossible for *any* combination or superposition of them to be other than linear. In other words, there can be no relation between any voltage and current within the box or *at its terminals* other than a simple linear one expressed graphically as a *straight line.*

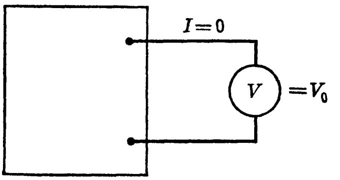


FIG. 9·12. Measuring open-circuit voltage.

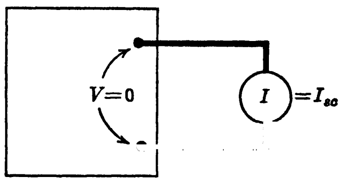


FIG. 9·13. Measuring short-circuit current.

Determining the V-I Characteristic. Let us measure the terminal voltage V_0 of the box with a potentiometer or high-resistance voltmeter as in Fig. 9·12 so that negligible current ($I = 0$) is permitted externally between the terminals. This is commonly known as an ***open-circuit test.***

Now let us measure the current between the terminals when we connect a zero resistance path between them as approximated by a good low-resistance ammeter (Fig. 9·13) so that $V = 0$ or no appreciable voltage remains between the terminals. This is commonly known as a ***short-circuit test.***

If we plot on a V-I graph these two points ($V = V_0$, $I = 0$) and ($V = 0$, $I = I_{sc}$) as in Fig. 9·14, we are assured by the superposition principle that a *straight line* through the two points gives the characteristic $V = f(I)$ for the boxed network at the two terminals.

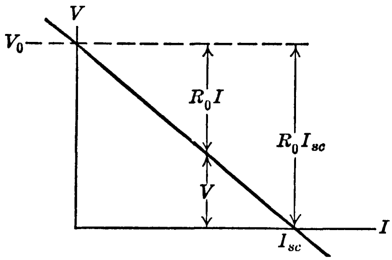


FIG. 9·14. Voltage-current characteristic for a Thevenin circuit.

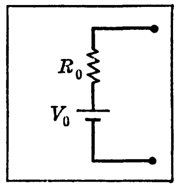


FIG. 9·15. A Thevenin equivalent circuit.

The Thevenin Equivalent Circuit. Now let us observe that the graph of Fig. 9·14 may be interpreted to represent simply a constant voltage V_0 minus a voltage drop expressible by R_0I where R_0 is a constant. The quantities are readily represented by a simple series circuit comprising a constant emf V_0 and a constant resistance R_0 as in Fig. 9·15. Because this simple circuit has the same V-I characteristic as the original given network, it can be enclosed in a duplicate box and substituted for the original for any purpose whatsoever that does not involve access to the contents of the box other than by the two available terminals. Such a substitute circuit is known as an ***equivalent circuit.***

The graph (Fig. 9·14) shows that when $I = I_{sc}$, $V_0 = R_0I_{sc}$ and that resistance R_0 is readily found from the test data by computing

$$R_0 = \frac{V_0}{I_{sc}} \qquad [9\cdot 19]$$

This procedure is familiar from computation of battery resistance which involves the same circuit of Fig. 9·15 to represent battery emf (V_0) and battery resistance (R_0).

Thevenin's theorem is not confined to the experimental or laboratory test procedure developed in the above. When the circuit for the network contents of the given box is known, it is possible to compute the

values V_0 and R_0 of the equivalent circuit and to simplify computation of a larger network which includes the complete boxed network or the substitute equivalent, as one of its component parts. The theorem, therefore, constitutes an artifice for simplifying the computation of certain types of network problems. Before proceeding with the application, let us consider how we may *compute* V_0 and R_0 when only the circuit diagram and values are provided, so that laboratory tests are not feasible.

Computing V_0 *and* R_0. The computation for V_0 involves the normal means for determining the voltage between two points in a network. In general, the Maxwell mesh method is best, but always tempered by a common sense inspection and analysis of the particular case.

The computation of R_0, unlike the experimental procedure, *does not involve finding the short-circuit current* I_{sc} for use in equation 9·19 but is obtained directly by another application of the superposition principle.

Let us look at a representative network such as may be in the box we have been discussing. This and its Thevenin equivalent are shown in Fig. 9·16*a* and Fig. 9·16*b* respectively.

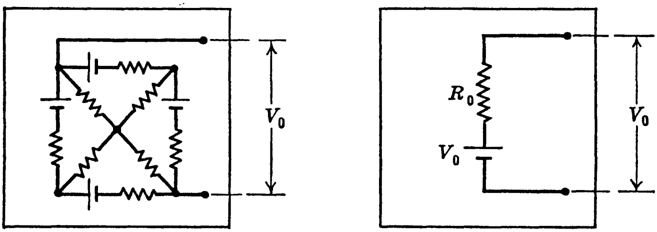


FIG. 9·16.

We have learned in the method of superposition that each emf in a constant-resistance network contributes to the current in any branch with strict independence from the contribution of any other emf. It follows that if *all emf's* are reduced by the *same* per cent, all currents and all *IR*'s *also* will be reduced by the same per cent. Because *any voltage* in the network is comprised *only* of these emf's and *IR*'s, it too must be reduced by the *same* per cent. This, of course, includes V_0 and requires that the emf V_0 of the equivalent circuit (Fig. 9·16*b*) be reduced by the same per cent in order to maintain the desired *equivalence*. In this manner let us reduce all emf's in the network until they become *zero*. Clearly $V_0 = 0$ results for the network and for its equivalent circuit. Because we have maintained the equivalence right up to the expiration of all emf's as a limit we are in position to assert that the

equivalence continues after all voltages are reduced to zero so that only resistances remain. We thus find that:

R_0 is simply the equivalent resistance of the given network when all emf's are reduced to zero.

When this all-resistance network is a series-parallel circuit, the solution for R_0 is obvious; but special methods such as the Y-Δ conversion, later considered in this chapter, may be required for more complicated cases.

Application of Thevenin's Theorem. Let us summarize the foregoing by a statement of Thevenin's theorem, as follows.

Any two-terminal network comprised only of constant emf's and resistances can be represented by an equivalent single emf and resistance in series.

This theorem was propounded by M. L. Thevenin in the latter part of the nineteenth century. It has long found favor in communication

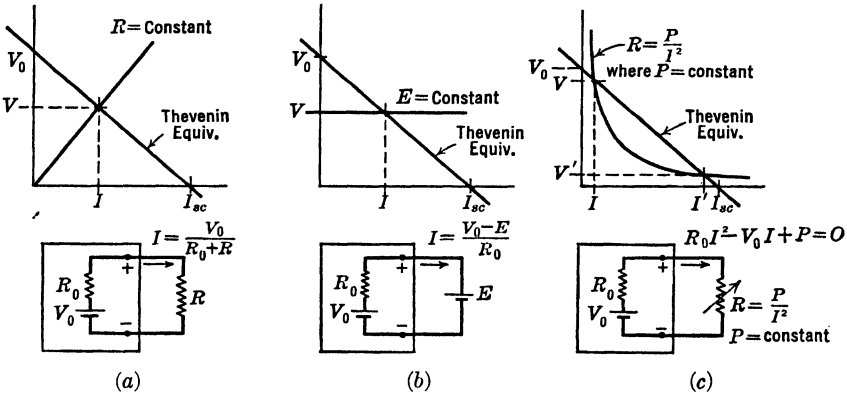


FIG. 9·17. Graphical representation of solution of selected circuits by Thevenin's theorem.

work and more recently for general circuit problems. It is particularly adapted to the computation of a network problem when only the solution for the unknowns of *one circuit element* is desired rather than a complete solution of the network. For this purpose all the network *except the one element* is "put in the box" and converted into a Thevenin equivalent comprising the V_0 and R_0 previously discussed. The one element is then connected across the terminals of the equivalent circuit which is readily solved. It is to be expressly noted that *there is no restriction on the character of the element which may be connected* ***externally*** to the terminals of the boxed network or its equivalent. It may be an emf, a

resistance, or any other device; it may either be constant or a function $V = f(I)$ without restriction.

This may be illustrated (Fig. 9·17) by plotting the V-I characteristic of the particular external element together with the linear V-I characteristic of the boxed network. The intersection of the two locates the particular values of V and I which will result from the connection of the particular circuit element across the terminals of the boxed network. Sometimes, as in Fig. 9·17*c*, there is more than one intersection but, as here, only one of these usually represents a stable condition of operation. If operation at the intersection shown for the smaller value of I is attempted, the slightest departure of the current or either voltage from the intersection values will produce an unbalanced voltage of such polarity as will aid the *departure* instead of the *return* of the current to intersection value. When the boxed network functions as the energy source, stability occurs only at an intersection where the slope of the V_0-I_{sc} line numerically exceeds that of the external V-I characteristic. Let us now follow through the solution of a typical problem which is advantageously solved by Thevenin's theorem.

Example. Given the circuit of Fig. 9·18*a*, it is desired to find $R = f(I)$. Clearly here we should find a Thevenin equivalent (V_0, R_0) for all the circuit

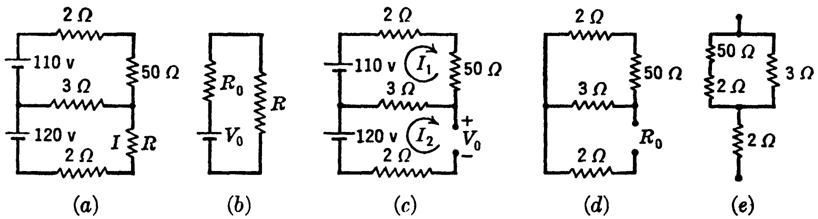


FIG. 9·18. Circuit representation of steps in solution of an electric circuit by Thevenin's theorem.

external to resistance R as in Fig. 9·18*b*. To find V_0 we must solve the circuit of Fig. 9·18*c*. This is readily obtained because I_2 is zero with the circuit open, as required, at V_0. Choosing the Maxwell mesh method, we write for the I_1 mesh,

$$(2 + 50 + 3)I_1 - 3(I_2 = 0) = 110$$

$$I_1 = \tfrac{110}{55} = 2.0$$

For the I_2 mesh,

$$(3 + 2)(I_2 = 0) - 3I_1 = 120 - V_0$$

$$V_0 = 120 + 3(I_1 = 2.0) = 126$$

The circuit for finding R_0 is given in Fig. 9·18*d*. Redrawn as in Fig. 9·18*e* it is readily shown to be a series-parallel combination which gives

$$R_0 = 2 + \frac{52 \times 3}{52 + 3} = 4.84 \text{ ohms}$$

Returning now to Fig. 9·18*b*, we find

$$V_0 = I(R + R_0)$$

$$R = \frac{V_0}{I} - R_0$$

$$R = \frac{126}{I} - 4.84 \text{ ohms}$$

Because of its simplicity there is a tendency to omit *drawing* the total equivalent circuit (Fig. 9·18*b*). Errors occur so frequently because of this that *special warning* against the omission seems advisable.

9·19. The Y-Δ Conversion. In the discussion of Fig. 9·2 it was brought out that the first concept of *equivalent resistance* is limited to

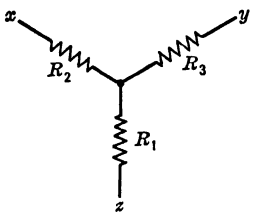


FIG. 9·19. Y-connected resistors.

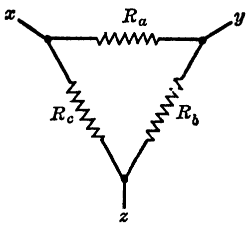


FIG. 9·20. Δ-connected resistors.

the special but very useful case of series and parallel connections. It is possible to extend this two-terminal equivalence to a case of three-terminal equivalence which, while quite as special as the former, is also quite as useful. This is called the Y-Δ conversion by reason of the appearance of the circuits as given in Figs. 9·19 and 9·20. The three resistances of either circuit can be so chosen that no measurement of resistance between the terminals x, y, z can disclose the identity of the one circuit from the other. The equations are established by considering only two terminals at a time and invoking the principle of superposition to insure that simultaneous use of all three terminals will not violate the results obtained from these equations.* From the equivalent resistance concept we write

For the wye:

$$R_{xy} = R_2 + R_3 \qquad [9\cdot20]$$

$$R_{yz} = R_3 + R_1 \qquad [9\cdot21]$$

$$R_{zx} = R_1 + R_2 \qquad [9\cdot22]$$

For the delta:

$$\frac{1}{R_{xy}} = \frac{1}{R_a} + \frac{1}{R_b + R_c} \qquad [9\cdot23]$$

$$\frac{1}{R_{yz}} = \frac{1}{R_b} + \frac{1}{R_c + R_a} \qquad [9\cdot24]$$

$$\frac{1}{R_{zx}} = \frac{1}{R_c} + \frac{1}{R_a + R_b} \qquad [9\cdot25]$$

Let the delta values R_a, R_b, R_c be given to find the *wye* resistances, R_1, R_2, R_3. These can be found in terms of the values measured at xyz by treating the wye equations as follows.

$$-R_{xy} + R_{yz} + R_{zx} = -(R_2 + R_3) + (R_3 + R_1) + (R_1 + R_2) = 2R_1 \qquad [9\cdot26]$$

$$R_{xy} - R_{yz} + R_{zx} = (R_2 + R_3) - (R_3 + R_1) + (R_1 + R_2) = 2R_2 \qquad [9\cdot27]$$

$$R_{xy} + R_{yz} - R_{zx} = (R_2 + R_3) + (R_3 + R_1) - (R_1 + R_2) = 2R_3 \qquad [9\cdot28]$$

From the delta equations we obtain

$$\frac{1}{R_{xy}} = \frac{R_a + R_b + R_c}{R_a(R_b + R_c)}, \quad R_{xy} = \frac{R_a(R_b + R_c)}{R_m} = \frac{R_aR_b + R_cR_a}{R_m} \qquad [9\cdot29]$$

$$\frac{1}{R_{yz}} = \frac{R_a + R_b + R_c}{R_b(R_c + R_a)}, \quad R_{yz} = \frac{R_b(R_c + R_a)}{R_m} = \frac{R_bR_c + R_aR_b}{R_m} \qquad [9\cdot30]$$

$$\frac{1}{R_{zx}} = \frac{R_a + R_b + R_c}{R_c(R_a + R_b)}, \quad R_{zx} = \frac{R_c(R_a + R_b)}{R_m} = \frac{R_cR_a + R_bR_c}{R_m} \qquad [9\cdot31]$$

Where

$$R_m = R_a + R_b + R_c \qquad [9\cdot32]$$

Substituting equations 9·29–9·31 in equations 9·26–9·28 we obtain

$$2R_1 = \frac{1}{R_m}(-R_aR_b - R_cR_a + R_bR_c + R_aR_b + R_cR_a + R_bR_c),$$

$$R_1 = \frac{R_bR_c}{R_m} \qquad [9\cdot33]$$

$$2R_2 = \frac{1}{R_m}(R_aR_b + R_cR_a - R_bR_c - R_aR_b + R_cR_a + R_bR_c),$$

$$R_2 = \frac{R_cR_a}{R_m} \qquad [9\cdot34]$$

$$2R_3 = \frac{1}{R_m}(R_aR_b + R_cR_a + R_bR_c + R_aR_b - R_cR_a - R_bR_c),$$

$$R_3 = \frac{R_aR_b}{R_m} \qquad [9\cdot35]$$

It will be noted that a certain *symmetry* exists in the above which assists the checking of errors.

The derivation of Δ-resistors in terms of Y-resistors is also useful and is left for the student to compute. The resultant relations are as follows.

$$R_a = \frac{R}{R_1} \qquad [9\cdot36]$$

$$R_b = \frac{R}{R_2} \qquad [9\cdot37]$$

$$R_c = \frac{R}{R_3} \qquad [9\cdot38]$$

Where R is the sum of the products of each and all combinations of two for the wye resistors, as follows.

$$R = R_1R_2 + R_2R_3 + R_3R_1 \qquad [9\cdot39]$$

To illustrate application of these relations, attention is directed to the bridge circuit of Fig. 9·21*a*. By converting the Δ of one loop to a Y,

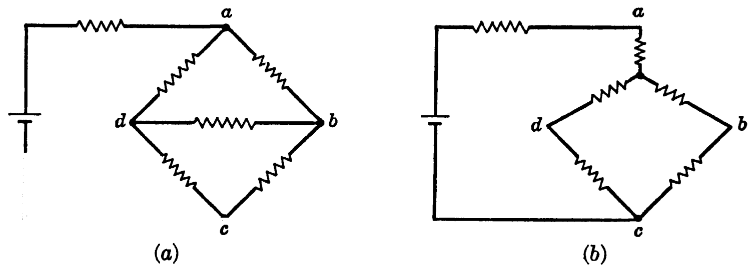


FIG. 9·21. Application of Δ-Y conversion for simplification of a bridge circuit.

the circuit degenerates into the series-parallel class as shown in Fig. 9·21*b* for easier solution.

Application of the reverse conversion is illustrated by the circuit of Fig. 9·11*a* reproduced in Fig. 9·22*a*. Using the Y-Δ conversion (Fig. 9·22*b*) two resistances become placed, one directly across each battery emf, while the third is readily seen to have only the difference of the two battery emf's impressed on it. When only the currents are unknown, the current in each resistance, as well as that in each battery, is readily computed.

It often happens that the Y trio of resistances appears more like a T as in Fig. 9·22*a*, and that the Δ trio appears more like a π as in Fig.

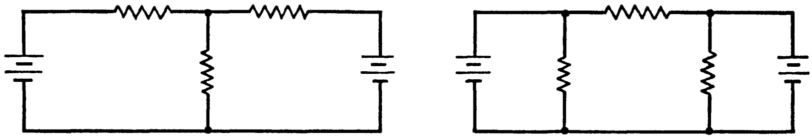


Fig. 9·22. Showing T-π form of Y-Δ conversion.

9·22*b*. For this reason the Y-Δ and Δ-Y conversions are quite as well known by the terms T-π and π-T conversion respectively. More often, however, the conversions are said to be *to the equivalent* π and *to the equivalent* T.

9·10. The Generalized Kirchhoff Laws. The network problems just considered have been restricted to direct current in the interest of comparative simplicity. It is beyond the scope of this text to consider the solution of a-c networks. Nevertheless it may well be noted here that the more general case of alternating current and the still more general case commonly termed *transients* are both based upon the same circuit principles just discussed in terms of their d-c application. The basic principles are embodied in Kirchhoff's laws.

Without becoming involved in the details of their application it is timely to learn how Kirchhoff's laws may be applied to circuits with *inconstant* emf's and currents, and to understand the symbolism necessary for unambiguous specification of these emf's and currents. This has already been initiated for emf's in Chapter VII. The notation for emf's of course concerns the Kirchhoff potential or loop relation $\Sigma V = 0$. Let us consider how the notation can be incorporated into the Kirchhoff potential law.

9·11. Kirchhoff's Law for Variable Potentials. Referring again to the surveying analog of the Kirchhoff potential relation, we have seen

that the algebraic sum of the rises and falls of electric potential taken around a closed traverse equals zero in the same sense that the algebraic sum of the ups and downs in elevation, or gravitational potentials, taken around a closed traverse, equals zero. It requires no elaborate argument to indicate clearly that the surveyor's traverse must close to zero even if the terrain he surveys is in a state of agitation such as might be represented by an ocean or by land during an earthquake. The important difference introduced by the inconstant terrain is the time element; the surveyor can no longer make his traverse leisurely but must have an instrument at each station around the traverse and take all readings both *simultaneously* and *instantaneously*. He might take numerous sets of data for a given traverse at various instants and find no two alike, nevertheless, *each* set should algebraically sum or "close" to zero.

In the same sense, the algebraic summation of electric potentials around any closed traverse must be zero *at any instant* for an electric circuit. How the electrical "terrain" may be varying is of no consequence; the potential difference between any two stations or circuit points may or may not experience reversals in polarity and may or may not be of a periodic nature. In recognition of this we may now write the Kirchhoff potential relation $\Sigma V = 0$ in the more general form $\Sigma v = 0$; the capital and lower-case letters signifying respectively *constant* and *variable* quantities in accord with the generally accepted convention. $\Sigma v = 0$ means that the algebraic summation of variable electric potentials around a closed traverse is zero *at any instant.*

9·12. Kirchhoff's Potential Law with Double Subscripts. Let us consider the circuit of Fig. 9·23 which, between the several circuit

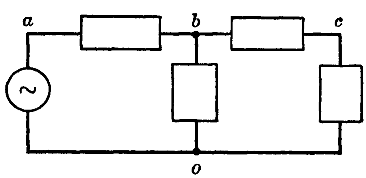


Fig. 9·23.

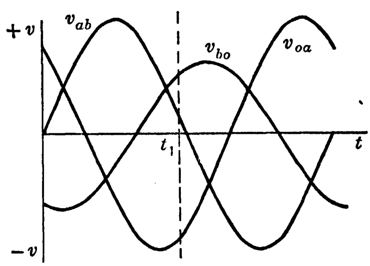


Fig. 9·24.

stations a, b, c, and o, has electric potentials represented on the oscillogram of Fig. 9·24.

To identify each voltage on the oscillogram with the two circuit stations between which it was measured, the identifying letter of each station is attached to the oscillographic curve concerned with these

stations. These appear as a double subscript to the voltage symbol, such as v_{ab} in Fig. 9·24 for the voltage between stations a and b in Fig. 9·23.

As discussed in Chapter VII, the subscripts serve not only to identify the circuit stations but also, by their *sequence*, serve to indicate how the algebraically (or graphically) *plus and minus regions of the oscillographic curve* are related to the *positive and negative polarities* of the station points on the *circuit*. In accord with the convention presented in Chapter VII: ***The circuit point designated by the first subscript has positive polarity for the algebraically plus values of $v = f(t)$ or for the plus (upper) regions of the oscillographic curve.***

We may then write the Kirchhoff potential equations unambiguously by using the double-subscript notation with this polarity significance of the subscript sequence. For the circuit of Fig. 9·23 these are

$$e_{oa} + e_{ab} + e_{bo} = 0 \qquad [9\cdot40]$$

$$e_{ob} + e_{bc} + e_{co} = 0 \qquad [9\cdot41]$$

The voltages for equation 9·40 are shown on the oscillogram of Fig. 9·24. It is to be noted that the equations are for instantaneous values which occur *simultaneously* at any instant such as t_1 on the oscillogram. Note also that *no minus signs* are used in these equations although certainly there is no instant when all the voltage magnitudes are to be summed arithmetically. At each instant the necessary minus values are produced automatically by the algebraic nature of the functions $e = f(t)$ and by the timing or phase spacing of them either on the oscillogram, like Fig. 9·24, or in the analytical equation form presented in Article 7·6 and further discussed here in Article 9·16. Clearly, if any curve on the oscillogram were upside down (180° out of phase), the algebraic sum at each instant such as t_1 would not be zero in accord with equation 9·40. It is the province of the *sequence* of each pair of double subscripts to insure against this possibility. Note that the correctness of these sequences is readily checked in any Kirchhoff voltage equation which is written, as here advocated, with voltages arranged in the order of encounter around the closed traverse and with no minus signs; the subscripts form a closed chain like *oaabbo* in equation 9·40. Evidently the writing of correct Kirchhoff voltage equations is quite as simple for a-c as for d-c circuits.

9·13. Kirchhoff's Law for Varying Currents. Kirchhoff's law for currents, like the voltage law, applies to the general case of varying currents as well as to direct currents. This is because the incompressible-like behavior of the electrons which comprise the currents of electric

circuits holds even for the vanishingly short lengths of time represented by an *instantaneous* measure of these currents. Within the limits of our present interest, at least, we may rest assured in the evidence that electrons still *behave as though incompressible* even for extremely rapid variations in current. We therefore generalize $\Sigma I = 0$ to $\Sigma i = 0$ where the latter means that the algebraic summation of all varying currents at a circuit junction is zero at any instant.

9·14. Kirchhoff's Current Law with Double Subscripts. Again let us consider the circuit of Fig. 9·23. We commonly use for current a double-subscript notation using the same station identifying letters that we used for voltage. The letters designate the portion or *branch* of a circuit (usually between junctions) throughout which the particular current exists. The sequence of the subscript letters indicates the direction of current which is taken to be algebraically plus. We shall consider the algebraically plus direction of current to be *from* the circuit station designated by the *first* subscript *toward* the circuit station designated by the *second* subscript. For junction b of the circuit of Fig. 9·23 we obtain the oscillogram of Fig. 9·25 and write for $\Sigma i = 0$:

$$i_{ab} + i_{ob} + i_{cb} = 0 \qquad [9\cdot42]$$

or

$$i_{ba} + i_{bo} + i_{bc} = 0 \qquad [9\cdot43]$$

If we arbitrarily agree to take the algebraically plus direction always *toward* the junction, it is evident that the second subscript will be that

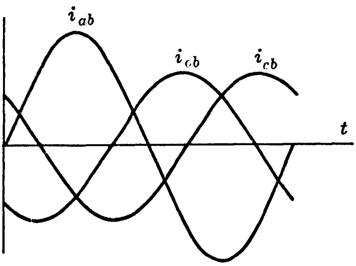


FIG. 9·25.

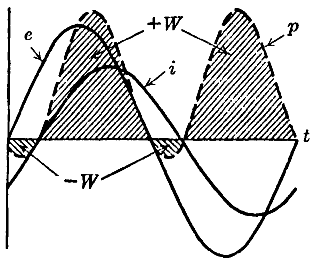


FIG. 9·26.

of the junction station for *every* current concerned. The equations are then written with the same simplicity and facility of checking that we found to distinguish the voltage equations, and need have no algebraic minus signs whatsoever. The absence of minus signs for currents is due to the same reason set forth for voltages. Minus signs will occur only when the subscript sequence is deliberately reversed from that utilized in writing the Kirchhoff equations, i.e., $e_{ab} = -e_{ba}$, and $i_{ab} = -i_{ba}$.

9·15. Subscripts for Loads and Sources. We have recognized early in this chapter the important definitive relations between direction of current and polarity of voltage for electric circuit loads and sources: current in a load is directed from positive to negative; current in a source is directed from negative to positive. This is equally true for *varying* currents and voltages. It is by no means uncommon to find that a given circuit element in an a-c circuit is alternately functioning as a load and as a source. Whether it is in the long run properly a load or source, of course depends on whether the *net* energy flow is into or out of the element in question. This is readily determined from the combined oscillogram of voltage and current taken simultaneously, as in Fig. 9·26, by plotting first the instantaneous power: $p = ei$. On the oscillogram this is simply a plot of the instantaneous products of e and i for a sufficient number of instants of time. The area under the $p = f(t)$ curve represents energy $W = \int p\,dt$. The power curve may have both plus and minus sections, i.e., above and below the time axis. The areas enclosed by these plus and minus sections represent flow of energy in opposite directions. But which areas represent load and which represent source energy?

For the plus and minus regions of the power graph to be identified as representing load or source power some unambiguous relation between the voltage and current oscillograms must be established. We have chosen to establish the *polarity* significance of the *voltage* subscript sequence as $+ -$ rather than $- +$ for the algebraically plus intervals. We have chosen to establish the *direction* significance of the *current* subscript sequence from first to second circuit point rather than the reverse. It follows that a voltage and current oscillogram of e_{ab} and i_{ab} taken between a given pair of circuit stations a and b ***will represent current from positive to negative as being algebraically plus,*** and that the *plus* power product must represent *load* power rather than power from a source. The area *above* the time axis therefore represents *load* energy and that *below* the axis represents *source* energy. If the plus or above-axis areas predominate, the circuit element ab is evidently functioning on the average as a load. The importance of this fact is frequently overlooked and deserves particular emphasis for this reason. In networks where both load and source elements occur in some profusion, failure clearly to identify each and distinguish it from the other will prove highly embarrassing. Such networks are not at all uncommon.

The need for distinction between load and source is readily indicated by considering the simple circuit for a single load connected to a single source as in Fig. 9·27.

For a short connection, the voltage e_{xa} and e_{yb} will be zero and we can write $e_{xy} = e_{ab}$. When current in the *load* is directed from a to b it is necessarily directed in the *source* from y to x, because the only path is the continuous loop $xabyx$. We must then write $i_{ab} = i_{yx}$. The power graph for element ab is then taken for $p = e_{ab}i_{ab}$ and will be predominantly *above*-axis (Fig. 9·28) or algebraically *plus*, and represents predominantly *load* power for ab. The power graph for element xy, if taken for $p = e_{xy}i_{xy}$, with the same subscript sequence for voltage and current,

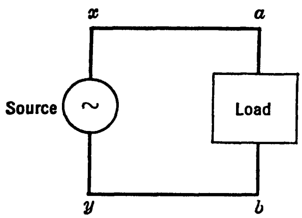


FIG. 9·27.

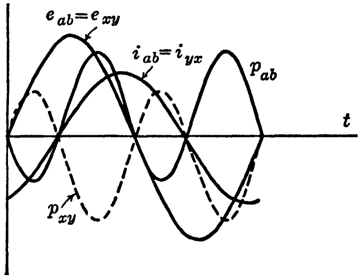


FIG. 9·28.

will be the same as for $p = e_{xy}(-i_{ab})$ and will be predominantly *below*-axis (Fig. 9·28) or predominantly *minus* load power as it should be for the *source* xy. The consequences of inattention to the subscript sequence of the e and i components of p are readily seen to be utter confusion. At the same time it must be noted that *the sequence of subscripts for p is meaningless*, i.e., $p_{ab} = p_{ba}$ and $p_{xy} = p_{yx}$. This occurs because the reversal of subscripts for *both* e and i, which p_{ba} denotes as compared to p_{ab}, reverses the sign of *both* factors of the product (power) and effects *no reversal* of the sign of the product p.

9·16. The Summation of Sinusoidal Quantities. When the voltages or currents to be summed in the application of Kirchhoff's laws are *sinusoidal* the process is conveniently executed in *analytical* form. It then becomes very helpful to develop a clear appreciation of three fundamental facts about sine functions of time as follows.

1. The summation of sine functions of the same frequency is a sine function of the same frequency.

2. The summation of sine functions of unlike frequency is not a sine function.

3. Any periodic function that might be obtained from an oscillogram can be expressed as the summation of a suitable number of sine functions of appropriate amplitude, frequency, and phase.

Further consideration of 2 and 3 will not be attempted here. Statement 1 however will be exploited in later chapters and is within the scope of immediate consideration.

Let us first consider one sinusoidal quantity such as a voltage e of Fig. 9·29 which in Chapter VII we found to be expressed analytically by the general expression

$$e = E \sin (\omega t + \phi) \qquad [9\cdot44]$$

From trigonometry we find that the sine of the sum of two angles provides the following expansion of equation 9·44.

$$e = (E \cos \phi) \sin \omega t + (E \sin \phi) \cos \omega t \qquad [9\cdot45]$$

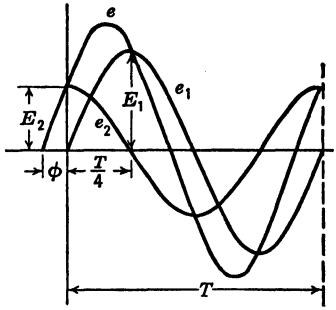


FIG. 9·29. Summation of sinusoidal voltages in quadrature.

Let us represent the coefficients of the two trigonometric functions of time by

$$E_1 = E \cos \phi \qquad [9\cdot46]$$

$$E_2 = E \sin \phi \qquad [9\cdot47]$$

Equation 9·45 then becomes

$$e = E_1 \sin \omega t + E_2 \cos \omega t \qquad [9\cdot48]$$

or

$$e = E_1 \sin \omega t + E_2 \sin \left(\omega t + \frac{\pi}{2}\right) \qquad [9\cdot49]$$

If we now let

$$e_1 = E_1 \sin \omega t \qquad [9\cdot50]$$

and

$$e_2 = E_2 \sin \left(\omega t + \frac{\pi}{2}\right) \qquad [9\cdot51]$$

then

$$e = e_1 + e_2 \qquad [9\cdot52]$$

It is clear that we have analyzed the voltage e into two component voltages e_1 and e_2 which are *sinusoidal* and of the *same frequency* ($f = \omega/2\pi$) *as* e. These component voltages are evidently 90 degrees out of phase or *in quadrature* and, as seen in Fig. 9·29, are readily found graphically, because when one has zero value the other has its greatest or amplitude value. The values of E_2 and E_1 are simply the values of e at time $t = 0$ and $t = \frac{1}{4}T$ respectively.

The operation just described of course is reversible so that any two sine waves *of the same frequency* which are in quadrature may be

combined into a single sine wave of the same frequency. The amplitude of this wave is found from equations 9·46 and 9·47 as follows.

$$(E \sin \phi)^2 + (E \cos \phi)^2 = E_1^2 + E_2^2$$

$$E^2(\sin^2 \phi + \cos^2 \phi) = E_1^2 + E_2^2$$

or

$$E = \sqrt{E_1^2 + E_2^2} \qquad [9\cdot53]$$

The phase angle is obtained from the same equations, as follows.

$$\frac{E \sin \phi}{E \cos \phi} = \frac{E_2}{E_1}$$

or

$$\phi = \tan^{-1} \frac{E_2}{E_1} \qquad [9\cdot54]$$

This summation of sine waves is restricted to waves in *quadrature.* The restriction is not insurmountable, however, if we are equal to a further diagnosis of the possibilities provided by the above two operations. We have found that we can operate to:

(*A*) Split any sine wave into a pair of sine wave components separated by 90 degrees, one with maximum value at the arbitrarily chosen time origin $t = 0$.

(*B*) Combine any two sine waves which are in quadrature and of like frequency into a single sine wave.

Let us now take *any* two sine waves of like frequency (Fig. 9·30):

$$e_a = E_a \sin (\omega t + \alpha) \qquad [9\cdot55]$$

$$e_b = E_b \sin (\omega t + \beta) \qquad [9\cdot56]$$

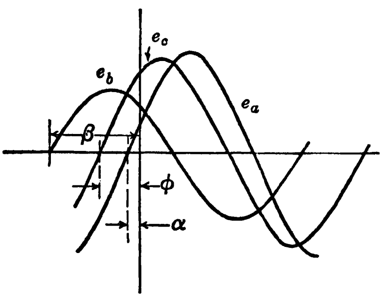


FIG. 9·30. Summation of sinusoidal voltages with phase difference.

and, by operation *A*, split each into the two quadrature sine wave components:

$$e_a = e_{a1} + e_{a2} = E_{a1} \sin \omega t + E_{a2} \cos \omega t \qquad [9\cdot57]$$

where

$$E_{a1} = E_a \cos \alpha \qquad [9\cdot58]$$

$$E_{a2} = E_a \sin \alpha \qquad [9\cdot59]$$

$$e_b = e_{b1} + e_{b2} = E_{b1} \sin \omega t + E_{b2} \cos \omega t \qquad [9\cdot60]$$

where

$$E_{b1} = E_b \cos \beta \qquad [9\cdot61]$$

$$E_{b2} = E_b \sin \beta \qquad [9\cdot62]$$

Now e_{a1} and e_{b1} are in phase and readily summed to give:

$$e_{c1} = e_{a1} + e_{b1} = E_{a1} \sin \omega t + E_{b1} \sin \omega t$$

$$= (E_{a1} + E_{b1}) \sin \omega t \qquad [9\cdot63]$$

Likewise e_{a2} and e_{b2} are in phase and readily summed to give:

$$e_{c2} = e_{a2} + e_{b2} = E_{a2} \cos \omega t + E_{b2} \cos \omega t$$

$$= (E_{a2} + E_{b2}) \cos \omega t \qquad [9\cdot64]$$

The voltages e_{c1} and e_{c2} are of like frequency and in quadrature so that, by operation (B), they are readily combined to give:

$$e_c = e_{c1} + e_{c2} = (E_{a1} + E_{b1}) \sin \omega t + (E_{a2} + E_{b2}) \cos \omega t$$

$$= E_c \sin (\omega t + \phi) \qquad [9\cdot65]$$

Where

$$E_c = \sqrt{(E_{a1} + E_{b1})^2 + (E_{a2} + E_{b2})^2} \qquad [9\cdot66]$$

and

$$\phi = \tan^{-1} \frac{E_{a2} + E_{b2}}{E_{a1} + E_{b1}} \qquad [9\cdot67]$$

Substituting the values from equations 9·58–9·62,

$$E_c = \sqrt{(E_a \cos \alpha + E_b \cos \beta)^2 + (E_a \sin \alpha + E_b \sin \beta)^2} \qquad [9\cdot68]$$

and

$$\phi = \tan^{-1} \frac{E_a \sin \alpha + E_b \sin \beta}{E_a \cos \alpha + E_b \cos \beta} \qquad [9\cdot69]$$

Since the quantities E_c and ϕ in equation 9·65 are independent of time, as shown by equations 9·68 and 9·69, the voltage e_c is a simple sine wave of the same frequency as its component sine waves e_a and e_b.

The summation of sine functions of the same frequency is a sine function of the same frequency, therefore, without restriction to waves in quadrature.

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1. Eshbach, "Handbook of Engineering Fundamentals," John Wiley and Sons, pp. 8-14–8-16.
2. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., pp. 2-20–2-26.
3. Dawes, "Electrical Engineering," Vol. I, McGraw-Hill Book Co., Third Edition, pp. 72–88.

QUESTIONS

9·1. Give the relation between polarity of voltage and direction of current in an electric circuit for (*a*) a load element and (*b*) a source element (or generator).

9·2. (*a*) State Kirchhoff's laws of the electric circuit.

(*b*) From what facts about electricity do Kirchhoff's laws come?

9·3. Explain why the Kirchhoff potential equation is expressed in terms of a closed *traverse* instead of a closed *circuit.*

9·4. Explain why it is recommended that the Kirchhoff potential equation be *not* expressed $\Sigma E = \Sigma IR$.

9·5. Explain why in general it is recommended that the traverse for a Kirchhoff potential equation be confined to a mesh boundary.

9·6. Outline a logical procedure for the solution of ordinary network problems by Kirchhoff's laws.

9·7. From the laws of Ohm and Kirchhoff derive the relations for finding the equivalent resistance for a sum of resistances connected in series and in parallel respectively.

9·8. What two distinct advantages does the Maxwell mesh method for solving electrical networks have over the direct Kirchhoff law method?

9·9. What restriction prevents the Maxwell mesh method from being universally applicable to electrical network problems?

9·10. What distinction is made here between the *principle* and *method* of superposition?

9·11. For what type of electrical network problems is the superposition method especially advantageous?

9·12. When applying the method of superposition to electric circuits with more than one source of energy, by applying one source at a time, should the *internal* resistance of these sources be left in the circuit when their emf's are considered to be zero? Explain.

9·13. Explain why the circuit of Fig. 9·11 cannot be solved directly by the method of superposition when the resistances R_1 and R_2, through which I_1 and I_2 flow respectively, are eliminated. Study the situation by solving for each current in terms of $E = E_1 = E_2$, $R = R_1 = R_2$, and R_3; then let R approach zero.

9·14. Explain how the principle of superposition is involved in Thevenin's theorem.

9·15. What is the purpose of visualizing circuit elements *in a box* for application of Thevenin's theorem?

9·16. By what argument is it established that R_0 in Thevenin's theorem is "simply the equivalent resistance of the given network when all emf's are reduced to zero"?

9·17. For what type of electrical network problem is Thevenin's theorem especially advantageous?

9·18. What restriction, if any, is placed on the nature of the circuit elements which may be connected externally to the terminals of the *boxed Thevenin equivalent* of a network section?

9·19. Sketch a typical boxed Thevenin equivalent circuit together with its *V-I* graph and explain the reasons for the shape of the graph.

9·20. Explain how the principle of superposition is involved in determining the relations among resistances for Y-Δ transformation.

9·21. Explain how the Y-Δ transformations are usefully applied.

9·22. An oscillograph curve of alternating current is labeled i_{ao}. Sketch the sec-

tion of circuit *ao* and explain with appropriate sketches what the curve signifies the actual current to be at several representative instants of time.

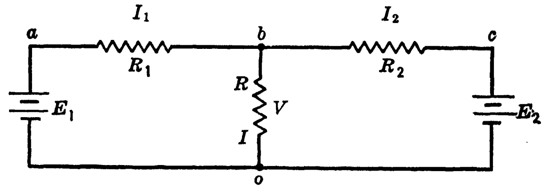
9·23. When *ab* is a *source* of power sketch the emf and current in phase and label each with the proper double subscripts.

9·24. Explain why, in double-subscript notation, it is possible and preferable to write either of Kirchhoff's laws with algebraic signs *plus* throughout the equation.

9·25. In an oscillogram such as given in Fig. 9·25 the relative direction of the three currents is changing from time to time through all possible combinations. Explain why this does not complicate the representation and identification of the currents in the double-subscript notation for each and every instant.

9·26. Expound three laws regarding the summation of sine functions and illustrate each by graphical example.

PROBLEMS



9·1. Given, for the above circuit:

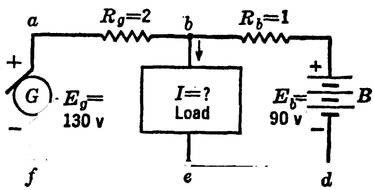
$$E_1 = 30 \text{ volts} \qquad R_1 = 2R_2 = 10R$$
$$E_2 = 25 \text{ volts} \qquad I = 20 \text{ amp}$$

Write the necessary and sufficient Kirchhoff equations and solve for I_1, I_2, R_1, R_2, R, V.

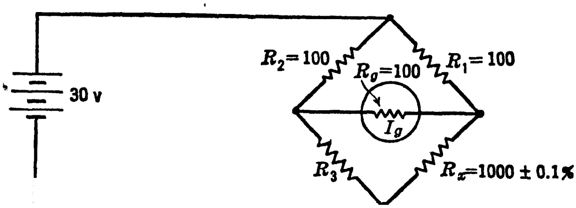
9·2. For the accompanying circuit determine the voltage E_{be} across the load and the current I_{be} taken by the load.

(*a*) When the battery is charging at a 10-amp rate.

(*b*) When the battery is discharging at a 10-amp rate.



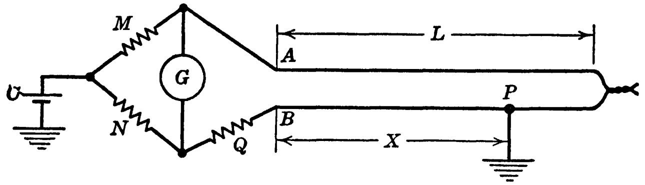
9·3. In a factory making precision resistors it is desired to select from production the 1000-ohm size within a tolerance of ±0.1 per cent. A measuring bridge is set up to do this with values as in the figure below. R_3 is adjusted for a balance ($I_g = 0$) at exactly 1000 ohms. A center zero galvanometer will be deflected one way or the other in accord with the discrepancy between the *particular* resistor R_x and 1000 ohms.



Compute the values $\pm I_g$ to be marked on the galvanometer scale as limits for the operator who is selecting the resistors.

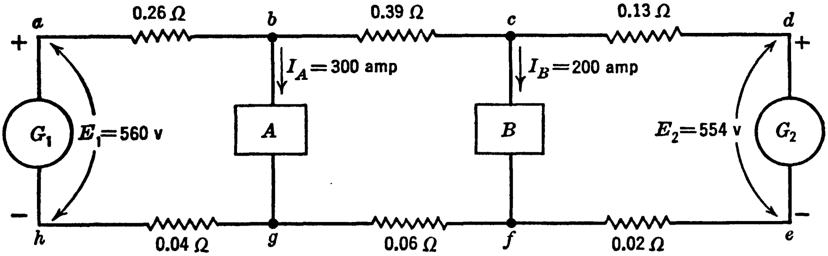
Note. Write the necessary Kirchhoff equations in literal form (no numbers) and solve by *determinants* for $I_g = f(E,R_1,R_2,R_3,R_x,R_g)$. Substitute numerical values and use log tables or other calculation more accurate than the slide rule, where required, to obtain I_g to *three significant figures.*

9·4. A Wheatstone bridge is often set up to locate an unwanted ground P, on a telephone line as shown in the accompanying figure. The far ends of the line are tied together while making the measurement and the near ends are connected to



the bridge at AB. If the two line wires have the same resistance per foot, derive the equation for the distance X out on the line to the grounded point P. Let the resistance per foot of line be denoted by r.

9·5. Using the potentiometer connected as shown in Fig. 5·3 an unknown voltage E_x was measured to be 2.0153 volts. It was later found that the potentiometer adjustment b had been set for a standard cell voltage $E_s = 1.01863$ volts whereas the cell voltage was actually 1.01883. If the potentiometer was otherwise properly operated, is it possible to compute the true value of E_x without repeating the measurement? If so, what is this value?



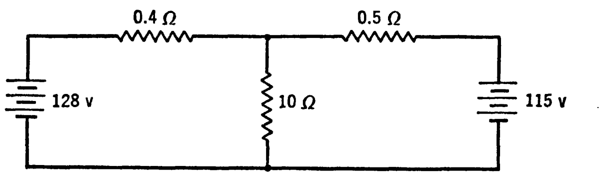
9·6. Determine for the above circuit:

(*a*) The current delivered by each generator.

(*b*) The voltages E_{bg} and E_{cf} across A and B respectively.

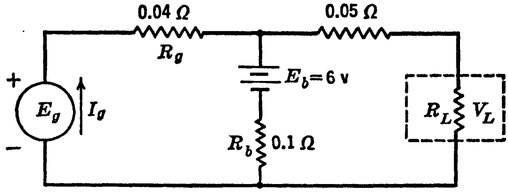
9·7. Find the power delivered by each battery using *each* of the following methods.

(*a*) Kirchhoff's laws.	(*c*) Thevenin's theorem.
(*b*) Superposition.	(*d*) The Y-Δ transformation of resistances.



Work carefully and accurately in each of the above so that you may compare results and judge which method is the most satisfactory for this particular problem.

9·8. The circuit in the accompanying figure is a simplified representation of an automobile electrical system comprising generator of emf E_g, battery of emf $E_b = 6$, and resistance 0.1, supplying lamps at V_L.



(*a*) Compute E_g required for $V_L = 6.5$ volts and $I_g = 20$ amp, using the Maxwell mesh method.

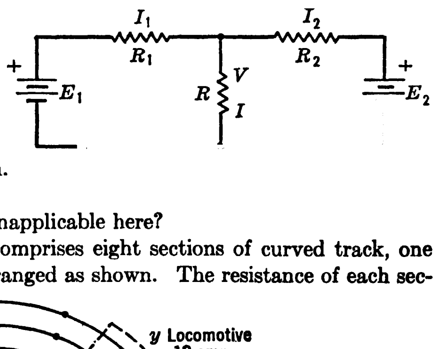
(*b*) If E_g varies between 6.0 and 9.0 volts what will be the greatest per cent deviation + and − of lamp voltage V_L from the original 6.5 volts? Assume constant resistance of lamp filament and use the method of superposition.

(*c*) It is frequently observed that the higher resistance of an aging battery or of a corroded terminal connection is accompanied by abnormal fluctuation of light with change in engine speed. Derive an expression for $dI_L/dE_g = f(R_b,R_g,R_L)$ and show cause for the effect.

9·9. Given, concerning the accompanying circuit:

$E_1 = 30 \qquad R_1 = 10R$

$E_2 = 24 \qquad R_2 = 5R$



When I is 20 amp find I_1, I_2, R_1, R_2, R, V.

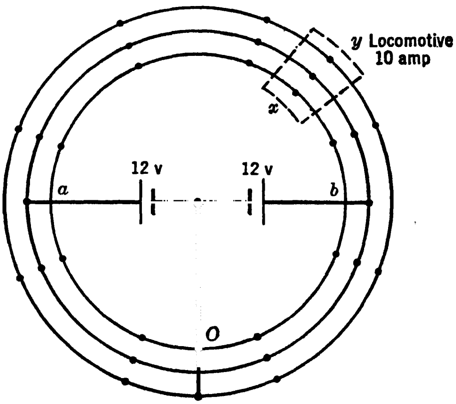
(*a*) Solve by Maxwell mesh method.

(*b*) Solve by method of superposition.

(*c*) Solve by Thevenin's theorem.

(*d*) Why is the Y-Δ transformation inapplicable here?

9·10. A miniature electric railroad comprises eight sections of curved track, one locomotive, and two sources of emf, arranged as shown. The resistance of each sec-



tion of rail is negligible compared with the joints which are 0.10 ohm for each rail joint. Using conventional symbols, redraw the circuit, combining the paralleled outside

rails, and combining the series joint resistances. By Thevenin's theorem find voltage V at the locomotive which is taking 10 amp.

9·11. Solve Problem 9·3 by the following methods:

(*a*) Maxwell's mesh method.

(*b*) Thevenin's theorem.

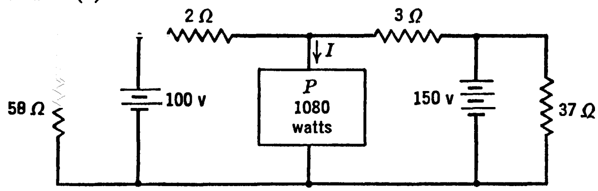
Discuss the relative efficiency of each of these methods and of Kirchhoff's laws as applied in Problem 9·3.

9·12. An electric railroad has a single track section 5 miles in length which is supplied with power from substations at each end of the section. The station at one end maintains a potential difference of 630 volts and that at the other end 530 volts between the third rail and the bonded track return. The total resistance of the line conductor and track return is 0.1 ohm per mile.

(*a*) At what point along the section would an electric locomotive taking 1000 amp receive a *minimum voltage* V between the third rail and the track? (Obtain $V = f$ (miles) by Thevenin's theorem.)

(*b*) What is the numerical value of the voltage V found in (*a*)?

(*c*) How much current will be supplied by each station when the locomotive is at the point found in (*a*)?



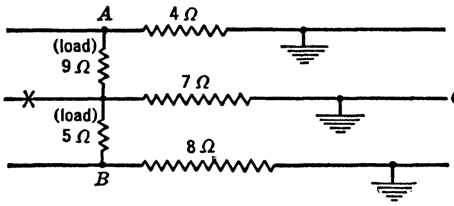
9·13. The power taken at load P in the above circuit is 1080 watts.

(*a*) By Thevenin's theorem compute current I through P.

(*b*) What is the lowest resistance that could be used at P for the given load?

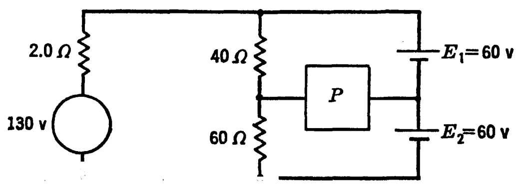
(*c*) Why is the Maxwell mesh method *not* suited to the solution of this problem?

9·14. After a violent storm a three-wire Edison circuit is in the condition shown in the accompanying figure. The neutral is open at X and all three wires are grounded



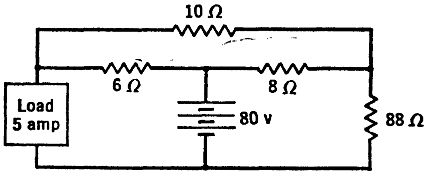
(short-circuited through ground) at such points as to make the resistances of the wires as shown. Compute the resistance between points A and B for the section of line shown. There are no grounds in the circuit except the three shown.

9·15. The batteries E_1 and E_2 in the accompanying circuit have negligible resist-



ance. Load P has a resistance expressed by $R = 6I$. Compute by Thevenin's theorem the power taken by P.

9·16. By applying the Δ-Y conversion reduce the accompanying three-mesh to an equivalent two-mesh network. Draw the circuit and compute:



(*a*) The value of each new resistance.

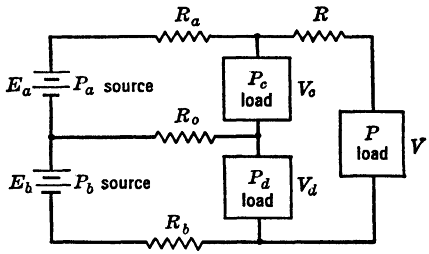
(*b*) By Thevenin's theorem applied to the new circuit, the power taken by the load.

9·17. (*a*) Given the circuit below and the listed values, solve by the Maxwell mesh method and find V, E_a, P_c, P_d.

(*b*) Explain why determinants are not useful in this problem.

(*c*) Check the conservation of energy according to power generated $(P_a + P_b)$ = power absorbed in the seven load elements.

R_a =	0.5	ohm
R_0 =	2.5	ohm
R_b =	1.0	ohm
R =	1.25	ohm
P_a =	18.0	kw source
E_b =	150	volts
V_c =	80	volts
V_d =	120	volts
P =	8.0	kw load



9·18. Given two currents as follows in the circuit of Fig. 9·23.

$$i_{ao} = 50 \sin (377t + 30°)$$

$$i_{bo} = 60 \sin (377t - 20°)$$

Evaluate I, ω, and α in

$$i_{co} = I \sin (\omega t + \alpha)$$

9·19. Given $i = i_1 + i_2$, where i_1 and i_2 are sine functions of the same frequency and with phase difference of 90 degrees, derive for this special case the amplitude and phase of i in terms of the constants of i_1 and i_2.

9·20. Given $i_1 = I_1 \sin \omega t$ and $i_2 = I_2 \sin (\omega t + \alpha)$, prove that the maximum value of $i = i_1 + i_2$ in *general* does *not* occur at instants when $i_1 = i_2$, i.e., where the graphs of i_1 and i_2 *intersect*.

9·21. Under what special conditions will the maximum value of $i = i_1 + i_2$ in Problem 9·20 occur at the instants when $i_1 = i_2$?

9·22. Given two sine functions:

$$i_1 = 10 \sin 377t$$

$$i_2 = 20 \sin (377t + 40°)$$

Using an 8 by 10½ sheet of graph paper ruled twenty to the inch:

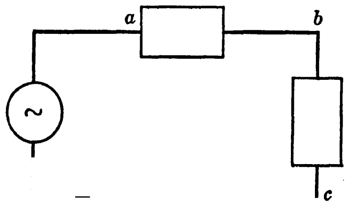
(*a*) Find $i = i_1 + i_2$ by *graphical* summation. Points taken at 30-degree intervals should suffice to locate curves i_1 and i_2 of good shape.

(*b*) *Compute* the amplitude I_m and phase angle α of the resultant current i, and check against the values obtained graphically.

9·23. Given the voltages:

$$e_{ab} = 100 \sin \left(377t + \frac{\pi}{4}\right)$$

$$e_{bc} = 100 \sin \left(377t - \frac{\pi}{4}\right)$$



(*a*) Draw a graph showing e_{ab}, e_{bc}, and e_{ac} plotted as functions of t, being sure to identify each clearly.

(*b*) Determine the minimum value of t for which $e_{ab} = -70.7$ volts.

(*c*) Determine by *inspection* of the graph the expression for the instantaneous voltage e_{ac} in the form $e_{ac} = C \sin (At + B)$ where A, B, and C are reduced to numerical values.

CHAPTER X

VARYING RESISTIVITY AND NONUNIFORM SECTION OF CONDUCTORS

The conductors studied in Chapter II were restricted to very limited geometry and conditions of operation. Although the bulk of practical problems are in this restricted class, equally practical *exceptions* are sufficiently numerous to warrant further study. We shall consider two classes of departure from the simplest circumstances.

1. Conductors where resistivity is not independent of the current.
2. Conductors where current density is not constant throughout the path of flow.

10·1. Varying Resistivity. That the electrical resistivity of materials in general is a function of *temperature* is familiar. The problems which we have considered so far involve only the simple relation $\rho_2 = \rho_1(T + t_2)/(T + t_1)$ which is derived from the concept of a temperature coefficient of electrical resistivity. It is not always feasible or desirable to determine these temperatures or the coefficients for them. In these cases it is commonly best to determine a curve of $E = f(I)$ from actual test conforming to the proposed conditions of operation of the particular apparatus in question.

The resistance of an incandescent lamp, for example, when operated at various d-c voltages as with a dimmer in theatrical work departs

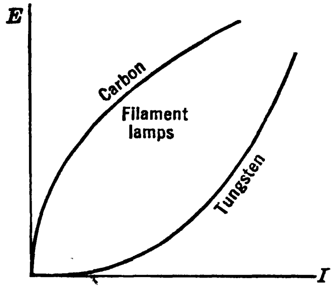


FIG. 10·1.

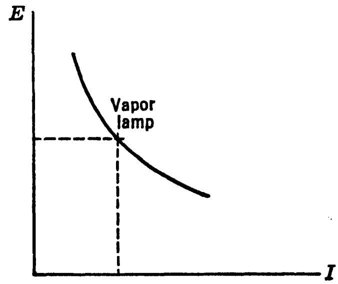


FIG. 10·2.

materially from the relation $\rho_2 = \rho_1(T + t_2)/(T + t_1)$ if T is considered to be constant. While it is a simple matter to measure the voltage and current so as to obtain $E = f(I)$ as in Fig. 10·1, it must be clearly under-

stood that the data may be of little value unless the measurement is made of the particular lamp located in the specific equipment (fixture, reflector, projector, etc.) and in the environment (ambient temperature, moving or still air, under water, etc.) conforming to actual operation. It is essential that time be allowed for the lamp to attain steady temperature before E and I are measured if data for the steady state are desired.

Other examples, where $R = f(I)$ is not constant, are common when nonmetallic conductors are concerned. Silicon carbide, carbon, General Electric's Thyrite (used in lightning arrestors), and numerous other materials may be cited. Electric conduction in gases characteristically involves resistances which decrease with increased current as in Fig. 10·2. The new fluorescent lamps and other vapor lamps, including so-called arc lamps, are of this character.

10·2. Varying Resistivity for Alternating Current. When the lamp or other varying resistance device is operated with rapidly varying or alternating current, the situation is further complicated. The thermal inertia of an incandescent lamp resistance, due to the specific heat (heat storage) of the filament, is sufficient to invalidate the graph of Fig. 10·1, for rapidly varying or alternating currents. The ultimate temperature corresponding to any instantaneous value of current is never attained before a new value of current sets a new ultimate of temperature. This is apparent when it is observed that the lamp filament operating on 60-cycle alternating current is always at a *luminous* temperature even though the current goes through zero. If this thermal inertia were sufficient there could be negligible variation in temperature and therefore in resistance. Lamps of higher wattages and lower voltages have filament proportions which approach this condition, but in general the variation of resistance is readily measurable. Figure 10·3 is a typical oscillogram of the current and voltage for a 115-volt, 15-watt incandescent lamp operating on sinusoidal 60-cycle voltage.

The oscillogram of Fig. 10·4 gives the voltage across and current through a 20-watt *fluorescent* lamp operating on a *sinusoidal* 115-volt supply. The lamp voltage is not sinusoidal because a *ballast* resistor or reactor is required in series with the lamp to control the current. Circuit elements which have $e = f(i)$ characteristics with negative slope (like Fig. 10·2) always require this ballast when they are supplied from either a d-c or a-c voltage supply of *constant* magnitude. For direct current let us assume the lamp to be operating with a given voltage and current at some point on the curve of Fig. 10·2. If any disturbance momentarily causes the current to increase, the *required* voltage as shown by the curve is reduced. The *supply* voltage now being more

than needed to sustain the current, an increase in current results. This further aggravates the condition initiated by the disturbance, and the current continues to increase uncontrolled. The instability is avoided by adding in series a circuit element such as resistance which has an

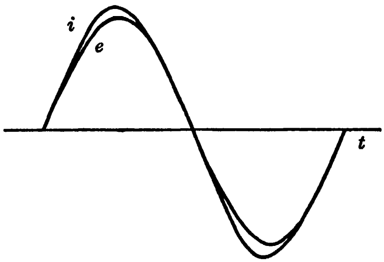


FIG. 10·3. Oscillogram for 15-watt incandescent lamp.

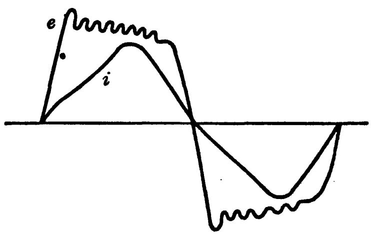


FIG. 10·4. Oscillogram for 20-watt fluorescent lamp.

$e = f(i)$ characteristic with sufficient positive slope to make the overall $e = f(i)$ characteristic of the two elements have a positive slope.

A curve of $e = f(i)$ for each lamp represented in Figs. 10·3 and 10·4 may be obtained from these oscillograms or, better, from new oscillograms taken with e and i applied to the vertical and horizontal oscillograph controls respectively. Figures 10·5 and 10·6 show such oscillograms corresponding to Figs. 10·3 and 10·4 respectively. These

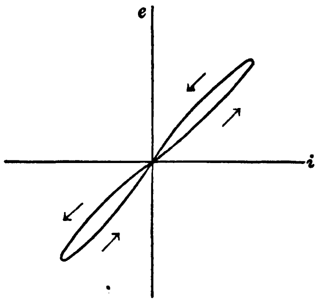


FIG. 10·5. Oscillogram for 15-watt incandescent lamp.

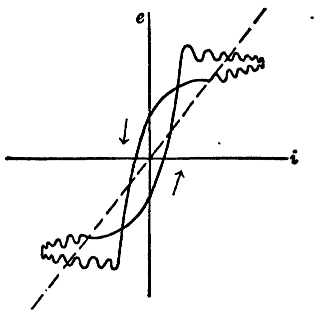


FIG. 10·6. Oscillogram for 20-watt fluorescent lamp.

$e = f(i)$ curves show the *dynamic* characteristic of each lamp as distinguished from the *static* characteristic of Figs. 10·1 and 10·2 and clearly indicate that they are but remotely related.

It is evident that the dynamic characteristic is not single-valued. This is no surprise if we have in mind that the conditions during current *in*crease differ from those during *de*crease. While the resistance may be

computed and plotted as an $r = f(i)$ curve it may suffice to observe in Figs. 10·5 and 10·6 that r is simply the slope of a line from the origin to any point on the curve.

10·3. When Resistance Falters. If we bear in mind that Figs. 10·5 and 10·6 represent but one specific pair of an infinite variety of possibilities it may be understood that there is no such thing as *the* dynamic characteristic for resistances which are not constant.

When resistance departs so completely from constancy as in Fig. 10·6, it is questionable whether the concept of resistance is then advantageous. Resistance is conceived by Ohm's law as a proportionality between voltage and current, $R = E/I$. When R becomes a function of I the proportionality ceases to exist. If the departure of R from constancy is great it becomes but wishful thinking to invoke artifices for prolonging the concept beyond its usefulness. In such case there is but one recourse: we simply revert to the original voltage and current of $R = E/I$, dismiss the R concept, and deal directly with $e = f(i)$ as exemplified in Figs. 10·5 and 10·6.

Under these conditions, especially when only graphical representation of $e = f(i)$ is feasible, the facility with which Kirchhoff's laws can be applied for solving networks of appreciable complexity is greatly curtailed. The semigraphical methods involved in such cases challenge ingenuity and draftsmanship. Fortunately the great majority of practical problems do not involve the combined difficulties of network complexity and grossly inconstant resistance. Deviations from constant resistance are commonly of such magnitude that a sufficiently accurate solution may be obtained by first assuming the resistance to be constant and then correcting for the effects of incremental departures from this constant value. It requires some experience to judge when this approximation may produce intolerable error.

10·4. Nonuniform Current Density. Somewhat as in hydraulic flow, there are various reasons why the velocity of flow or the corresponding current density may not be uniform throughout the cross section of a conductor. Even when the current is unvarying, lack of uniformity in the material or dimensions of the conductor or in the Ohm's law potential along the various parallel filamentary current paths through the conductor will cause nonuniform current distribution. When the current is varying, effects analogous to those due to the inertia of water and the elasticity of the conducting pipe are encountered.

10·5. Skin Effect.* A common case of the above, *skin effect,* arises whenever the current in a conductor is *varying with time,* either pulsat-

* "Standard Handbook for Electrical Engineers," pp. 2-98–2-102, 4-27–4-29, 12-476, 14-52, 15-127.

ing or alternating. It is characterized by a density of current (or velocity of electron drift) which varies *spacially* from a maximum at the periphery of the conductor section to a minimum, or in some cases even reversed flow, at the center. The causative agent is the electromagnetic induction of emf discussed in Chapter V. In explanation let us consider Fig. 10·7 which represents a cylindrical conductor as though comprised of thin concentric tubes of thickness dr and radius r. A current directed along each tube produces magnetic flux *only external* to the tube and not within the bore of the tube. Because the current is varying with time, the flux is varying likewise and will induce emf in any conductor which it links. Because that portion of current even in the smallest tube produces flux, it is evident that there is flux throughout the conductor. Now observe clearly that only that flux which is *external* to a given tube can *link* with the tube and induce an emf in it. It follows that the *smaller the tube*, the *more flux is linked* with it and the *more emf* is induced in it. As will later be discussed, these induced emf's are of such polarity as to oppose the current flow. Thus we account for the lower current density at the axis as compared with the periphery of the conductor, which characterizes the phenomenon.

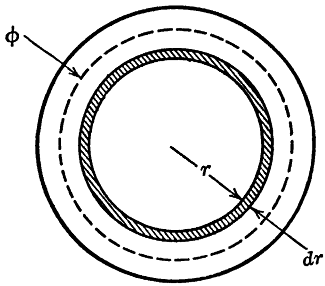


FIG. 10·7.

While skin effect is basically an *emf* distortion it is convenient and usual to express the data in terms of the equivalent per cent increase in the conductor *resistance*. Skin effect of course is a function of frequency and wave form of current as well as size and shape of conductor. Values for sine waves and conductors of ordinary shapes are common handbook data.

At power frequencies of 25 to 60 cycles the effect is of importance only in the larger conductors or conductors containing magnetic material, such as the stranded, steel-cored, aluminum cable designated A.C.S.R. (aluminum cable steel reinforced). At communication frequencies, especially radio frequencies, skin effect is a major factor and special conductor designs are commonly employed. One of these, called ***Litzendraht*** (abbreviated, Litz) wire, subdivides the conductor into several smaller component conductors, insulated from one another and transposed or scrambled in their relative position from one cross section to the next so that any one strand will eventually occupy in succession all possible positions from axis to periphery. Hollow tubing and flat or ribbon conductors are also used, especially where rigidity is important.

10·6. Nonuniform Section. When successive sections along a conductor vary in area, the density of current must vary accordingly so that the current flow through each section will maintain the requisite constancy imposed by its incompressible character. The computation of resistance then becomes a problem in integration. Because the elemental resistances dR are electrically in *series* the integration is simply the summation of the dR's, which according to Davy's law becomes

$$R = \int \frac{\rho}{A}\, dl$$

When $A = f(l)$ is expressible in analytical form the process is obvious and there are some simple cases, such as the radial current flow in a disk or the flow along a tapered tube, where the computation may be useful.

It must be kept in mind quite clearly, however, that electric current, somewhat like a *low*-velocity water current, pursues a partially streamlined path which may not conform to the apparent path provided by the conductor. This is particularly to be expected when a change in sectional area or dimension is comparatively abrupt. A few of these have been handled successfully by analytical process but in general a *graphical* procedure is to be recommended.

10·7. Mapping Current Flow. Starting with a scale drawing of the conductor, two sets of lines are drawn, the one representing paths of current flow, the other indicating the locus of points of like potential. The latter lines or sectional planes are called ***equipotential lines*** or surfaces. They are important because they must everywhere be *normal* to the lines of flow and consequently serve to assist in shaping or mapping the flow lines. This perpendicularity requirement arises from the basic tendency of electric current to seek the shortest path between successive planes of different potential. Our discussion will be confined to cases which can be represented in two dimensions, i.e., where the third dimension is uniformly constant.

Let us first consider a conductor which changes from section A to section B with a reasonably tapered transition as in Fig. 10·8.

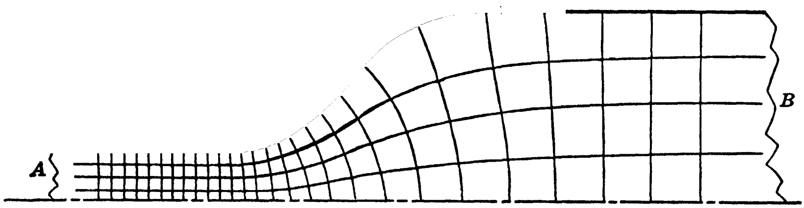


Fig. 10·8.

While it is true that an infinite number of lines of either flow or potential may be drawn it is desirable to draw only such lines as will represent a uniform increment of the variable concerned, i.e., the spacing of the equipotential lines to represent a fixed increment ΔE from one line to the next line, and the spacing of the current lines to represent a fixed increment of current ΔI between adjacent lines. It is helpful further to choose such increments ΔE and ΔI as will make the spacing of potential lines equal to the spacing of flow lines for any given point or infinitesimal $de \times di$ area in the diagram. This means that the *finite* $\Delta E \times \Delta I$ figure should be a *square.* This materially assists in judging the correctness of the map and in properly locating the lines. That the squares may not be of uniform *size* is apparent when it is realized that increase in sectional area of conductor will reduce current density dI/dA which is measured by the spacing of the flow lines. Unless the potential gradient is reduced in proportion to the current density, the squares will become rectangles. Fortunately, in a medium of uniform resistivity it is so reduced, as the following relation indicates.

Given:

$$E = RI \quad \text{or} \quad \Delta E = R\Delta I$$

and

$$R = \rho\frac{l}{A} \quad \text{or} \quad R = \rho\frac{\Delta l}{\Delta A}$$

we find

$$\Delta E = \rho\frac{\Delta l}{\Delta A}\,\Delta I$$

or

$$\frac{\Delta E}{\Delta l} = \rho\frac{\Delta I}{\Delta A}$$

and

$$\frac{dE}{dl} = \rho\frac{dI}{dA}$$

Thus, *so long as the conductor material is of uniform resistivity* ρ, the potential gradient dE/dl is proportional to the current density dI/dA and the square *shape* of $\Delta E \times \Delta I$ will prevail even though the *size* may not. The linear size of each square will of course be *inversely* proportional to the current density or to the potential gradient in the region of any given square. In other words, the density or closeness of the lines is a measure of the density of current for the flow set of lines and of the gradient of potential for the potential set.

Some of the squares will be *curvilinear* squares and more difficult to identify. It then becomes necessary to subdivide these into smaller

squares which approach the more familiar rectilinear shape as the subdivision is continued and thus demonstrate the correctness (or otherwise) of the original assumption of square shape.

This is better shown in the more difficult problem of Fig. 10·9 which represents a sharp right angle bend in a conductor of rectangular section. Subdivision of the standard square is shown near the line of symmetry SS'. Of course not all cases possess symmetry, but when present it is clearly of considerable assistance in constructing the map.

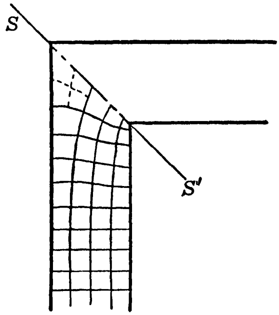


Fig. 10·9.

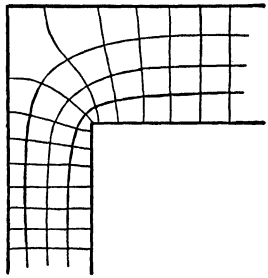


Fig. 10·10.

An unsymmetrical case is given in Fig. 10·10 which combines the right angle bend with change of section. Obviously considerable practice is required to attain any proficiency in producing accurate maps. A sharp pencil (not a pen), good paper, a soft eraser, and some degree of patience materially promote a successful project of this character.

When the map is completed, the resistance of the conductor is computed by observing that each square of whatever size represents a fixed resistance $R = \Delta E/\Delta I$ where ΔE and ΔI are the values already chosen as a measure of the spacing of the respective sets of lines. By counting the squares of R's in series and in parallel the computation of the total equivalent resistance is a problem in elementary physics.

It may be observed here that this attack is useful not only in the solution of problems of electric conduction but also in electrostatic and magnetic problems which are even more commonly encountered in present-day engineering practice. These will later be discussed and the importance of the method indicated.

10·8. Refraction of Current. It has been mentioned in the foregoing that electric current seeks the path of least resistance. When current flows from a conductor of one resistivity to another of different resistivity, the path of least resistance involves a refraction suggestive of that experienced with light. So long as the path of current flow is normal to the boundary surface between the two conductors no effect

results, but when the situation is as shown in Fig. 10·11 the flow is distorted. The basic relation is

$$\frac{\tan \alpha_A}{\tan \alpha_B} = \frac{\rho_B}{\rho_A}$$

where α_A and α_B are the respective angles of incidence for the current flow in material of resistivity ρ_A and ρ_B.

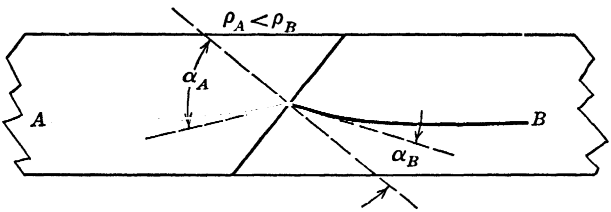


FIG. 10·11. Refraction of current at junction of dissimilar conductor materials.

It may be observed that, contrary to the above implication, there is little analogy between light and electrical conduction; the equation for refraction of light involves the ratio of sines rather than tangents, concepts of potential and potential gradient are lacking, etc.

10·9. Ground Connections. The study of electrical engineering is frequently pursued to a rather advanced state before it is disclosed that the *grounding* of an electric circuit is beset with serious practical difficulties which commonly make the assumption that a point on an electric circuit is readily brought to earth potential disturbingly naive. Ground connections are usually made with the intent of preventing the appreciable departure of some part of an electric circuit from earth potential, thereby reducing the possibility of unexpected potentials hazardous to life and property. The emergency for which the ground connection is intended to function commonly involves very large current through the connection to or from the earth. Because it is impractical to make the connection to earth sufficiently extended to provide perfect contact, an IR potential always accompanies the current. It is clearly possible for this potential to be very large if the *ground resistance* is not made sufficiently low to accommodate the emergency current. A ground resistance as low as 0.001 ohm of course would have 100 volts potential for 100,000 amp. The design of ground connections and the measurement of their resistance are problems of considerable challenge which today engage some of the best available research facilities.

Ground resistance involves both varying resistivity and nonuniform conductor section. For an introductory study we may best confine

our attention to the latter (and usually more influential) feature. Depending on the nature of the soil, the climate, and the class of ground connection required for a particular case, the structure may vary all the way from one or more driven rods to an extensive array of copper conductors buried like a mat at considerable depth. Consideration of the driven rod will suffice for our purpose here.

10·10. Determining the Ground Resistance of a Driven Rod. Let the rod be represented, as in Fig. 10·12a, by a cylinder A of diameter D_a

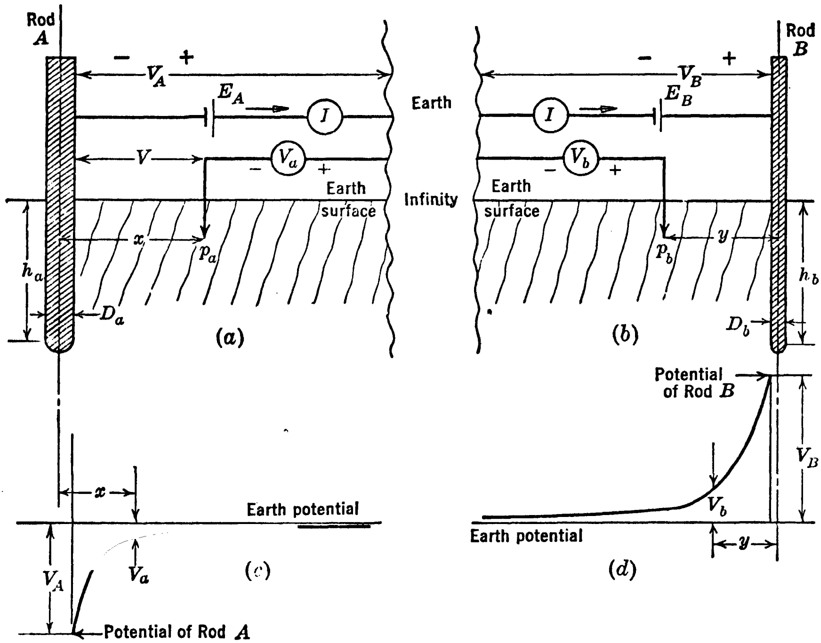


Fig. 10·12. Analysis of voltages for grounding rod carrying current.

with the end rounded off into a hemisphere, and driven to depth h_a. A source of emf E_A, connected between the rod and a distant ground assumed at infinity, produces current I which is assumed to flow to the rod in paths represented by the radii of both cylinder and hemisphere extended toward infinity in earth of uniform resistivity ρ.

A *probe* at any ground point p_a distant x from the *rod* will be at some *negative* potential V_a with respect to *earth*. If $\int I\,dR$ is taken from $x = \infty$ to $x = D/2$, V_a attains value V_A as graphed in Fig. 10·12c.

Since V_A is the potential of *rod* with reference to *earth* (infinity), the ground resistance of the rod is simply $R_A = V_A/I$.

In the actual measurement of ground resistance it is necessary to drive another rod such as B in Fig. 10·12b to carry current I into the ground. Rod B is driven to depth h_b and is assumed to have shape similar to rod A.

If we compute for rod B the potential V_b of any *probe* point p_b with respect to *earth* (infinity) in the same manner as for A we obtain a graph as in Fig. 10·12d where V_B is the *positive* potential of rod B with respect to *earth.*

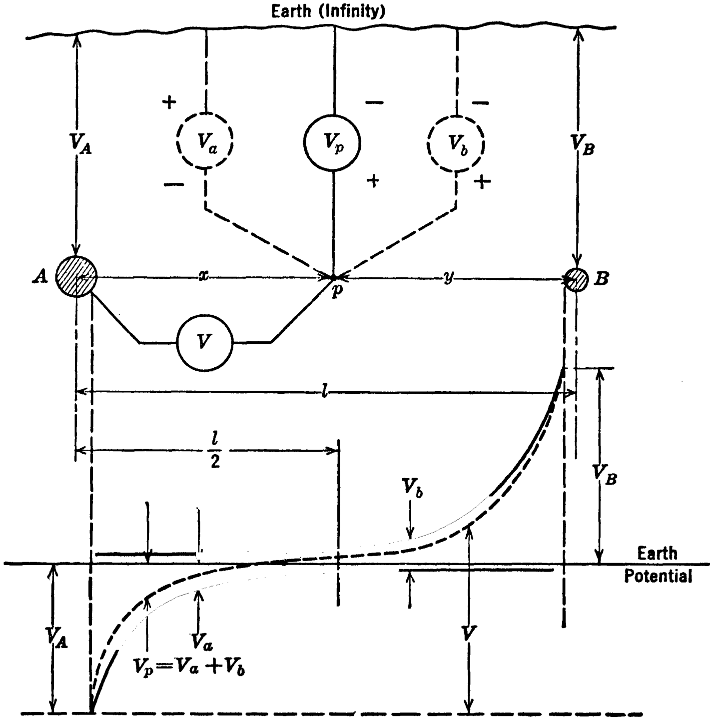


FIG. 10·13. Measurement of ground resistance of a driven rod.

Clearly it is impossible to provide the *infinite* separation between rods A and B suggested by Figs. 10·12a and b. On the basis of constant resistivity we may invoke the aid of *superposition* and conceive that rods A and B are oriented so as to have in fact a *finite separation* l as represented in the plan view of Fig. 10·13a without abandoning the reference of each to *earth* (infinity). Points p_a and p_b may now be merged into point p which by superposition will have, relative to *earth,* potential $V_p = V_a + V_b$ which is shown in Fig. 10·13b by the dotted curve.

If there were any simple means for locating the potential probe at the point where $V_p = V_a + V_b = zero$, voltage V_A could be readily measured from rod A to probe p located at this point, and $R_A = V_A/I$ obtained. Since this is not feasible it becomes necessary to locate B *far* enough from A so that $V_p = V_a + V_b$ will remain *near* zero for some distance around the point of *true* zero.

It is sometimes possible to locate point B at considerable distance by utilizing a power transmission line for the lead to the remote ground connection.

Probe p is commonly driven about midway between A and B. Several instruments designed especially for the measurement of ground resistance by this method are available. They are constructed somewhat along the lines described for the two-coil ohmmeter in Chapter IV but supply alternating rather than direct current for the ground current in order to minimize electrolytic effects.

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QUESTIONS

10·1. Consider an example where the resistance of a conductor varies with temperature according to a nonlinear graph of $R = f(t)$ and explain how to find the heat energy generated by a constant current during a given time while the temperature is changing. Take into account both the *storage* of heat energy in the conductor and the heat *dissipated* by various means.

10·2. Explain why a *ballast* is required to operate a fluorescent lamp from a potential of fixed magnitude.

10·3. In the discussion of skin effect the terms *density of current* and *velocity of electron flow* were used synonomously. Will drift velocity of electron flow bear the same relation to current density in aluminum as in copper? Explain.

10·4. Describe the construction of Litz wire and explain how it combats skin effect. Given the same amount of copper for constructing a conductor of Litz or of thin-wall tube design, which will have the lower resistance to high-frequency current? Explain with appropriate sketches.

10·5. Why are flow-potential maps for irregular conductors scaled to form squares rather than rectangles?

10·6. From the principles expounded in this chapter sketch a flow-potential map for a section of bare current-carrying conductor immersed in a poorly conducting liquid of infinite extent so that a small amount of its total current leaks from the conductor to the liquid.

10·7. Explain why a rod driven into the ground may not suffice to *ground* an electric circuit.

PROBLEMS

10·1. A length of copper tubing which weighs 0.48 lb per ft is to be annealed by heating with electric current. As a first approximation compute the time required for a current density of 6000 amp per sq in. to heat the tubing from 20° C to 700° C, assuming no loss of heat, no change in dimensions, and temperature coefficient of resistance $\alpha_{20} = 0.00393$ to hold up to the desired 700° C.

10·2. A 2-ft length of No. 6 copper conductor passes through a wall from temperature $t_1 = 70°\ F$ to $t_2 = -20°$ F. Compute the resistance of the conductor, *assuming uniform temperature gradient.* Discuss the possibility of the temperature gradient affecting the potential drop along the conductor.

10·3. (*a*) Derive the function $V_a = f(I,\rho,h_a,x)$ for the ground rod in Fig. 10·12*a*.

(*b*) Find V_A and R_A for $x = D/2$ in Fig. 10·12*a*.

(*c*) Derive the function $V_b = f(I,\rho,h_b,y)$ for the ground rod in Fig. 10·12*b*.

(*d*) Derive the expression for probe potential $V_p = V_a + V_b$ relative to *earth.*

(*e*) Given: A, a 1¼-in. rod driven 10 ft into the ground.
B, a ¾-in. rod driven 3 ft into the ground.
$l = 25$ ft.

With probe p arbitrarily located midway between A and B (Fig. 10·13) compute the per cent error which results from assuming that the ground resistance of rod A is $R'_A = V/I$, where V is the potential of probe relative to rod A and the true resistance is $R_A = V_A/I$. Observe that, because $V - V_A = V_p$ (Fig. 10·13*b*), the per cent error is simply $100V_p/V_A$ when V_p is the probe potential relative to *earth* at the midpoint $x = y = \frac{1}{2}$.

(*f*) Repeat (*e*) for $l = 50$ ft instead of 25 ft.

(*g*) Is the greater spacing in (*f*) really effective in reducing error? What minimum spacing would you recommend when ordinary portable instruments with about 1 per cent error are used?

CHAPTER XI

MAGNETICS

11·1. Magnetism. It was little more than a century ago that *magnetism* was definitely shown to be associated with an electric *current.* The deflection of a compass needle by a current in a nearby parallel wire was discovered by Oersted in 1820. It is of interest that the discovery was no surprise to Oersted; he had long *sought* this effect and further suspected that static, galvanic, magnetic, and chemical phenomena might *all* be closely related. We now recognize all magnetic phenomena to be but one aspect of *electron gymnastics* and accept Oersted's suspicions as commonplace fact.

11·2. The Magnetic Field. If the space environing a source of magnetism is explored with a tiny compass needle, the direction of the magnetic effect throughout the space can be charted. Faraday, a contemporary of Oersted's, very helpfully introduced the use of lines variously called *lines of force, lines of induction,* and *lines of flux* for this purpose. Such lines, as shown by iron filings on paper, are familiar to all at an early age. The lines are closed or continuous lines which, in terms of the compass needle, represent closed traverses of the tiny compass pursued continuously in the indicated direction of the needle. Similar lines are used to represent light flux, strains in mechanically stressed bodies, etc. Clearly, the lines have *no physical reality* even though they may seem to be taken so seriously in the computation of fields as to imply reality.

Quite as for these analogous cases, the concept is extended to indicate magnitude as well as direction of the magnetic effect. This is accomplished by spacing the lines in proportion to the strength of the magnetic effect in each and every part of the picture. *Density* of flux then denotes intensity of magnetic effect. It follows that the total number of lines represents the total magnitude of the magnetic effect (or flux) —indeed, it is commonly used as a unit of *measure* for magnetic flux.

11·3. Flux Units. *Quantity of magnetic flux* is quite universally represented by symbol ϕ. The unit of measure has long been the *line* or *maxwell* with the convenient enlargement to *kiloline* (10^3) in accord with the well-known prefix practice of the metric system. (While a unit kilomaxwell would seem equivalent to the kiloline it is seldom used.)

In 1935, with the adoption of the MKS (meter-kilogram-second) system by the International Electrotechnical Commission, a new unit of magnetic flux, the ***weber,*** equal to 10^8 lines was given official sanction. Under various names it had long been advocated and used by eminent scholars in the field of electrical engineering. For the derivation of relations involving magnetic flux the weber is the natural unit in the so-called practical system of units and will be used here invariably for this purpose. In the shop and in the realm of design, however, the kiloline has become so well established that it is unreasonable to expect that it may be abandoned for purely academic reasons in favor of the weber. In recognition of this the kiloline will here be used when adherence to practice is deemed to outweigh the logic of consistency for the academically rational weber unit. Again, let it be emphatically clear that: ***All basic relations, derivations, and formulas involving magnetic flux will here be in terms of webers unless definitely specified otherwise.***

Density of magnetic flux is symbolized by $B = d\phi/dA$. The MKS unit, of course, is the ***weber per square meter.*** Because the square meter area is rather large for most practical situations, there has been some tendency to use the *weber per square centimeter* and the *weber per square inch.* So long as it is observed that these do not fit basic formulas without conversion constants, there seems to be no serious objection to their use, especially to the use of the weber per square centimeter which is a decimal part of the MKS unit.

The CGS unit of magnetic flux density is the line per square centimeter which is commonly known by the simpler name ***gauss.*** The kiloline per square centimeter or ***kilogauss*** is frequently used because of its convenient size for many purposes.

The English unit is the line per square inch or the ***kiloline per square inch.*** The present practical usage in the United States is divided largely between gauss or kilogauss and the line per square inch or kiloline per square inch.

Consistent with the policy indicated above for flux, all basic relations, derivations, and formulas involving magnetic flux density will here be in terms of *webers per square meter* unless definitely specified otherwise.

11·4. The Magnetic Circuit. Flux lines never cross and are endless or closed. This continuity of the lines of a magnetic field suggests an analogy with the lines of current flow in an electric circuit and gives rise to the concept of a *magnetic circuit.* The lines in this case, however, *do not represent flow* as for current because the magnetic field represents a state of strain rather than a kinetic phenomenon. Magnetic *strain* or flux is produced only by a magnetic *stress.* The source of magnetic stress is the electric current which also comprises a *closed circuit.* The

electric circuit *links* with the magnetic—and usually more than once, as exemplified in the ordinary multiturn coil so common in electrical apparatus. These stress sources, while not so obviously connected *into* the magnetic circuit, as batteries are connected into an electric circuit, nevertheless are parts of the magnetic circuit and generate the flux-producing magnetic potential in much the same way as the battery of an electric circuit generates (chemically) the current-producing electric potential.

11·5. Magnetic Potential. Magnetic potential is another of the family of potentials discussed along with electric potential or emf. Consequently magnetic potential, magnetomotive force or mmf $\mathfrak{F}$ is fairly analogous to electric potential, electromotive force, or emf E. As noted above, mmf's have their source in electric currents which commonly link the magnetic circuit in the form of a coil. When current I flows in each of N linking turns of a coil, the effective total linking current is $\mathfrak{F} = NI$. That is, current I flowing in N turns produces the same mmf as NI amperes flowing in *one* turn which has the same dimensions as the complete N-turn coil. Mmf, in the practical or MKS system, is measured in ***ampere-turns.*** In the CGS electromagnetic system, the unit is the ***gilbert*** which is equivalent to $(0.4\pi)^{-1}$ ampere-turn and is *smaller* than the ampere-turn.

The magnetic field is conceived to store energy—magnetic energy. Like emf, mmf is an energy derivative, $\mathfrak{F} = dW/d\phi$. It represents, *per unit of flux*, the energy which goes into a growing magnetic field or out of a decaying field. In MKS units, ***one joule per weber equals one ampere-turn.*** This relation is used in computing magnetic energy $dW = \mathfrak{F}\, d\phi$ and will be referred to later.

11·6. Permeance and Permeability. It is convenient to think of mmf and flux as cause and effect, respectively, in much the same sense as we think of emf and electron displacement, or pressure and mechanical displacement. The medium or body participating in these phenomena is characterized by the ratio of cause and effect which obtains for the particular body. By analogy with the electrical circuit we write as follows in MKS units:

Circuit		Cause	Effect	Cause/Effect	Effect/Cause
Electric	Symbol	E	I	R	G
	Unit	volt	ampere	ohm	(mho)
Magnetic	Symbol	$\mathfrak{F}$	ϕ	$\mathcal{R}$	$\mathcal{P}$
	Unit	amp-turn	weber	amp-turn/weber	weber/amp-turn

It is more usual in magnetic circuits to employ the permeance $\mathcal{P} = \phi/\mathfrak{F}$ rather than the reluctance $\mathcal{R} = \mathfrak{F}/\phi$. No names for the units of either permeance or reluctance have as yet received official or

general acceptance. Occasionally the henry is ascribed to permeance but this is not consistent with the well-established use of the henry for the unit of inductance, as will later become evident in the discussion of inductance.

In the same sense that resistance depends upon material and dimensions according to $R = \rho(l/A)$, we find that magnetic reluctance and its reciprocal, permeance, depend upon the material or medium in which the magnetic field is created, and upon the dimensions of the field or flux path. We express this for the reluctance as $\mathcal{R} = (1/\mu)(l/A)$ or, more commonly, for the permeance as $\mathcal{P} = \mu(A/l)$. Factor μ is called the ***permeability*** of the medium of the flux path and is the permeance of a unit dimensioned ($A = 1$, $l = 1$) piece of the medium, i.e., the magnetic material. As for resistivity ρ, it is convenient to visualize the unit dimensioned piece as a cube. The cube will be expressed in meters, centimeters, or inches, according to the system of units employed. We shall use the *meter cube*, of course, in the MKS system.

11·7. Magnetic Potential Gradient. If the relation involving permeability $\mathcal{P} = \mu(A/l)$ be substituted in the relation $\phi = \mathcal{P}\mathfrak{F}$ we obtain the following.

$$\phi = \mu \frac{A}{l} \mathfrak{F}$$

$$\frac{\phi}{A} = \mu \frac{\mathfrak{F}}{l}$$

$$B = \mu H$$

The flux density symbolized by B has already been discussed but the quantity $H = \mathfrak{F}/l$ is new here. Length l is taken along the lines of magnetic flux so that $\mathfrak{F}/l$ signifies change in magnetic potential (rise or fall) per unit length of flux path. Quite as in the case of the electric circuit, where change in electric potential per unit length of current path V/l, is electric potential gradient, so $H = \mathfrak{F}/l$ is ***magnetic potential gradient.*** This is numerically equivalent to the quantity called *magnetizing force* which possibly is more familiar to the reader. The usefulness of the potential gradient concept will become evident in applications to be discussed presently.

The MKS unit of H is the ***ampere-turn per meter.*** The CGS unit is the gilbert per centimeter, known since the International Electrotechnical Congress of 1935 * as the ***oersted.*** Observe that the oersted is

* Until this action of the IEC the oersted was used for the CGS unit of reluctance (one oersted = one gilbert/maxwell). This seemingly unnecessary switch of the oersted will explain the confusion in the use of the name which the reader may encounter in literature.

$1/0.4\pi$ amp-turn per centimeter and is *smaller* than the ampere-turn per centimeter. In the United States the oersted shares favor with the ampere-turn per inch, and the MKS unit (ampere-turn per meter) has not yet appeared commercially.

11·8. Rationalized versus Unrationalized Units. For the basic magnetic circuit relation, $\phi = \mathcal{P}\mathcal{F}$, we have established that the MKS units for ϕ and $\mathcal{F}$ are respectively the weber and the ampere-turn while the CGS units are respectively the line or maxwell and the gilbert. It has already been pointed out (Chapter VI) that the MKS weber is the natural unit of flux involved in the very important process of electromagnetic induction. The weber is thus established in terms of the incontroversial volt and second units: one weber = one volt-second.

The MKS unit for $\mathcal{F}$ has been given here as the *ampere-turn.* Unlike the weber unit for ϕ this does not have an official status because the International Electrotechnical Congress has not been able to agree whether the $\mathcal{F}$ unit should be the ampere-turn or the gilbert. The same controversy has existed for over a half century in the CGS system although in this system the gilbert is the official unit.

The controversy begins with the work of Oliver Heaviside in 1892 when he pointed out that there were advantages to using plain ampere-turns instead of gilberts for $\mathcal{F}$; that this could be done by assigning the unavoidable 0.4π factor to *permeance* $\mathcal{P}$ in $\phi = \mathcal{P}\mathcal{F}$ instead of to *magnetic potential* $\mathcal{F}$. Since the units for A and l of $\mathcal{P} = \mu(A/l)$ necessarily are unaffected, the 0.4π factor is in turn assigned to *permeability* μ so that the permeability μ_0 of free space becomes 0.4π instead of unity. For free space it seems of little consequence whether the 0.4π is assigned to $\mathcal{F}$ or $\mathcal{P}$. The advantage of Heaviside's concept comes from the fact that, for magnetic materials, values of permeability and permeance seldom enter directly into computation because curves of B versus H are preferable when $\mu = f(B)$. The 0.4π factor is thus frequently avoided if assigned to μ rather than to the units of magnetic potential and potential gradient. Heaviside considered this to be the more reasonable or rational of the two alternative choices for the assignment of the 0.4π and thus introduced the rationalized or 0.4π-less units of magnetic potential and potential gradient. Thus we speak today of *rationalized* and *unrationalized* units, in accord with his views and nomenclature.

A large amount of data on the permeability of various materials is available in the standard CGS units. To convert this into rationalized units might seem impractical for the advantage gained. Fortunately this is avoided by introducing the concept of ***specific permeability*** in line with the familiar specific gravity, specific heat, etc. A *specific* value, in this sense, is a *ratio* of one quantity to another of the same dimension,

the latter being an arbitrarily chosen standard of reference. For permeability the standard of reference naturally is μ_0, the value for free space. The permeability μ of any given material then is

$$\mu = \mu_s \mu_0$$

where μ_s is the specific permeability of the given material. Because the units for measure of μ and μ_s are taken to be identical, μ_s is *dimensionless* and independent of the system of units used for μ and μ_0. Now note that because μ_0 is *unity* in standard CGS units, the numerical values of μ and μ_s in these units must be identical. It follows that the permeability values already available in the standard *un*rationalized CGS units may simply be *relabeled* μ_s and utilized as *specific* permeabilities in *any* system of units, including the MKS, and regardless of rationalization.

Because the author favors the rationalized system we shall here use for the *MKS* system *only the rationalized units.* It is in line with this policy that the MKS unit of magnetic potential has already been arbitrarily referred to here as the ampere-turn.

The MKS unit of permeability has dimension $\mu = B/H$ or $\dfrac{\text{webers/meter}^2}{\text{ampere-turn/meter}}$ = *webers/meter-ampere-turn.* There is for this no generally accepted one-word equivalent. For free space $\mu_0 = 0.4\pi \times 10^{-6} = 1.257 \times 10^{-6}$ weber/meter-ampere-turn.

11·9. Values for Specific Permeability. Comparatively few materials are known to have a specific permeability μ_s differing considerably from unity. Some substances have values of μ_s *greater* than unity only by about 10^{-6} and are termed ***paramagnetic.*** Other substances have μ_s values *less* than unity by similar amounts and are said to be ***diamagnetic.*** The μ_s of these materials differs too little from unity to concern the engineer professionally.

A very limited group of paramagnetic materials has values of μ_s markedly greater than unity. These are ferrous substances, principally steels or alloys of iron, and are termed ***ferromagnetic*** materials. Detailed consideration of these materials is not within the scope of an elementary study of established fundamentals but it is by no means to be construed as unimportant to the electrical engineer. The student should eventually acquire considerable familiarity with both the magnetic and mechanical properties as well as the metallurgical aspects of the ferrous alloys which have been and are being developed especially for the electrical engineer.

For some years the *silicon steels* ranging from 1 per cent to 4 per cent silicon have occupied first place among commercial magnetic materials. Other alloys under coined and trade names have attained real impor-

tance. The Permalloy of the Western Electric Co. and the Westinghouse Electric and Manufacturing Company's Hipernik merit particular mention. Values of specific permeability μ_s range from only about 1.01 for so-called *nonmagnetic* steels in the austenitic state, up to several thousand for silicon steels and to maxima of the order of 1,000,000 for Permalloy. It is to be noted that μ_s depends not only upon the chemical *composition* of materials but also upon the chemical and mechanical *state* of the material, its *temperature*, and even its *present* and *past magnetic states* as represented in units of flux density B.

11·10. *B–H* Curves. It will be found of far-reaching and varied consequence that permeability μ_s for ferromagnetic materials is *not constant* but is a function of flux density B. This function $\mu_s = f(B)$ is best expressed in the form of a curve of μ_s versus B because it is not amenable to expression in practical mathematics. Typical curves for the usual materials are available in several handbooks and in the publications of the manufacturers. Note that these are but *typical* curves; in practice it is usual to run curves on specimens of the actual material to be used in a particular case. Considerable variation is likely from batch to batch of what is presumably the same material.

It will become apparent presently that, for the computation of magnetic circuit fluxes and mmf's, the permeability curve is not so convenient as a curve of B versus H or "B-H curve" as it is commonly termed. Because $B = H\mu$, it is evident that the same basic test data on the material is expressible in either form. The steel manufacturers commonly publish both curves. Although the English kiloline per square inch, and ampere-turn per inch units, are sometimes used for these data, there is clearly a preference for the metric *kilogauss* and *oersted* units (Figs. 11·1 and 11·2).

Because the meter unit of length is inconveniently large for expressing the dimensions of all except the largest heavy power equipment in electrical engineering, there is some question whether the MKS units may readily find favor for the special purpose of computing magnetic circuits in place of the well-established CGS rationalized or unrationalized units. In view of this these computations (Chapter XII) will be conducted in units according to the present practice and the typical B-H curves given there are so dimensioned. It should be noted that the kilogauss or kiloline per square centimeter unit used there for B is equivalent to one-tenth of a weber per square meter or a deciweber per square meter, that the ampere-turn per centimeter used for H is equivalent to 100 ampere-turns per meter or a hecto-ampere-turn per meter. The scales there used are in this sense decimal multiples of the MKS units and are readily converted if desired.

The curves of Figs. 11·1 and 11·2 are of typical materials and in units commonly used by the manufacturers. The scale for H is often made logarithmic. This horizontally stretches the curve out in the lower range of values and compresses it in the higher range to provide more uniform readability of the curve throughout a longer range of values. The principal disadvantage of this for our purpose here is that the shape of the curve, thus distorted, makes it more difficult to see how the various curves approach saturation.

The log-log plot of Fig. 11·2 provides no improvement in readability of B-H values but makes possible the direct reading of *permeability* values from the B-H curve. This is an interesting method for introducing what is effectively a third axis of values but it is little used.

11·11. Mapping Magnetic Fields. In Chapter X the mapping of lines of current flow and of equipotential was discussed in some detail. The analogy between lines of current flow and lines of magnetic flux, and between equipotential lines for electric and for magnetic potential is so close that all the principles set forth regarding the mapping of the electrical case apply equally well to the magnetic case. From the practical point of view, there is this important difference—the configuration of magnetic circuit structures is as commonly *irregular* as that of electric conductors is *regular*. Consequently mapping magnetic fields is a usual rather than an occasional need. Furthermore magnetic materials afford no such extreme range in the values of μ as we find for ρ. Thus we have no magnetic "insulators," and the amount of magnetic flux outside of the desired path is commonly so great that careful consideration and computation of it is required. For this purpose, especially in the more complicated structures, mapping is commonly conceded to be the most effective tool now known.

11·12. Refraction of Magnetic Fields. Because magnetic flux so commonly is not entirely confined to the magnetic path provided for it, attention to the refraction phenomenon is important. This relation for magnetic flux is exactly what might be expected from previous experience with refraction and is as follows, where α_1 and α_2 are the angles made with the normal to the boundary by flux lines in media of permeability μ_1 and μ_2 respectively.

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2}$$

When steel having μ_s of the order of 1000 is concerned, so that $\mu_s/\mu_a = 1000$, it should be clear that lines in air ($\mu_s = 1$) must have close to zero angle of incidence α_a. Flux lines in air, therefore, as they enter or leave good magnetic steel must be drawn practically

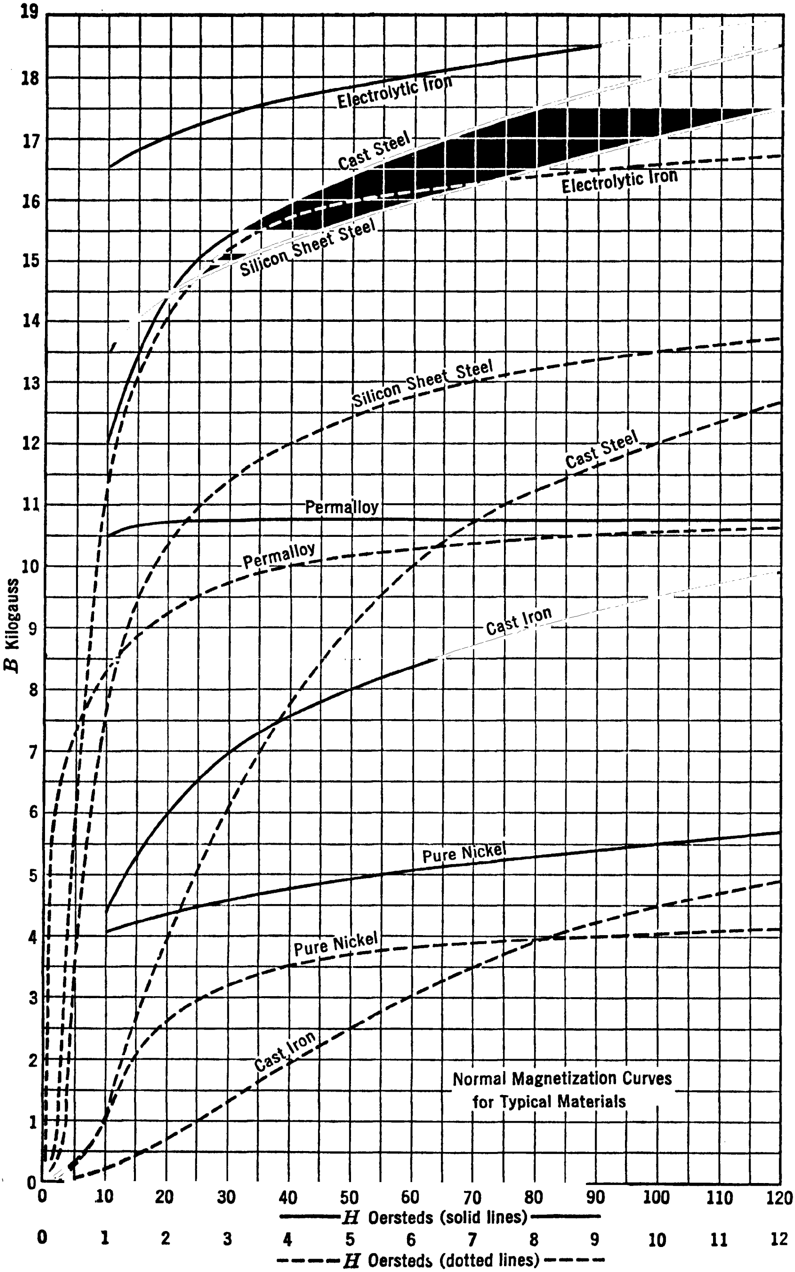

FIG. 11·1. *B-H* curves of typical materials.

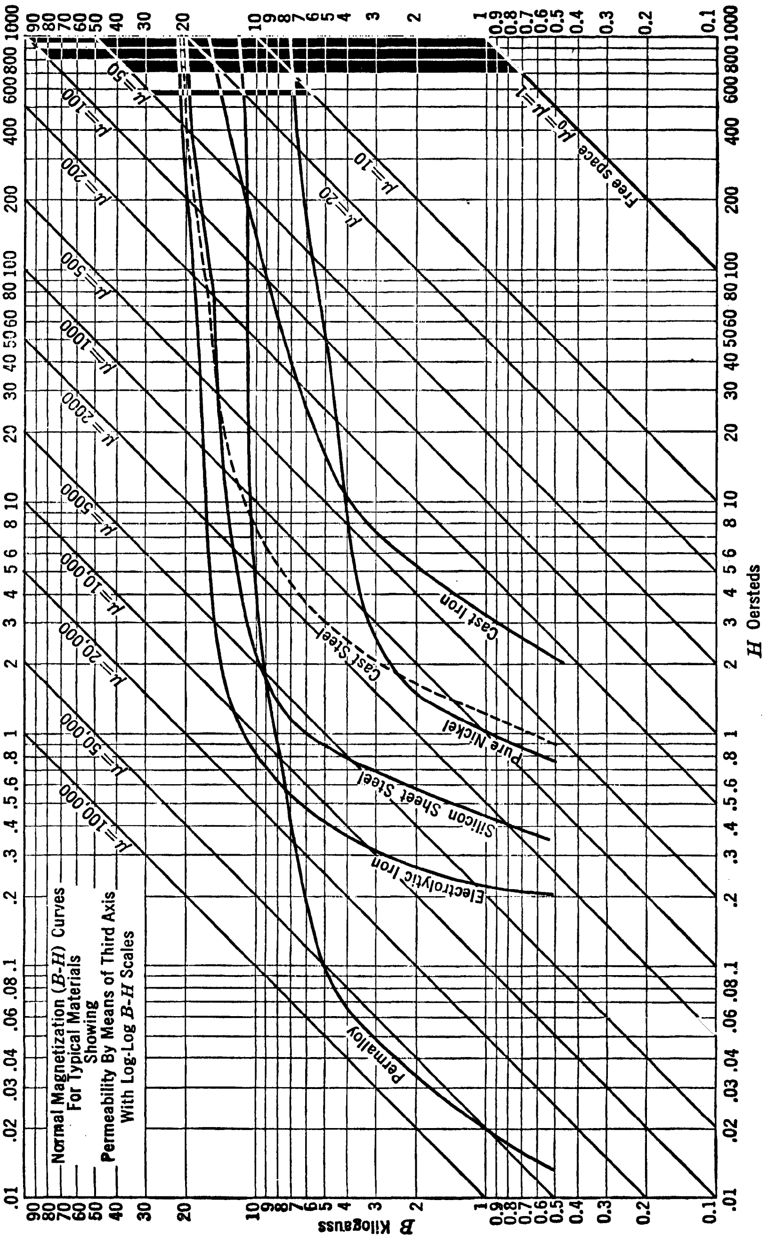
Normal Magnetization (*B-H*) Curves
For Typical Materials
Showing
Permeability By Means of Third Axis
With Log-Log *B-H* Scales
B Kilogauss
H Oersteds
Free Space $\mu = 1$
Cast Iron
Cast Steel
Pure Nickel
Silicon Sheet Steel
Electrolytic Iron
Permalloy
$\mu = 10$
$\mu = 20$
$\mu = 50$
$\mu = 100$
$\mu = 200$
$\mu = 500$
$\mu = 1000$
$\mu = 2000$
$\mu = 5000$
$\mu = 10{,}000$
$\mu = 20{,}000$
$\mu = 50{,}000$
$\mu = 100{,}000$

Fig. 11·2.

perpendicular to the surface for all angles of incidence within the steel up to 85 degrees or more.

As before indicated, mapping of this kind requires more time for practice than for learning the principles.

11·13. Superposition of Fields. When μ is a *constant*, as for air, the principle of superposition is applicable to $B = \mu H$ and it is possible to map the field for each of two or more sources of mmf independently and to find the resultant field by combining the individual fields instead of the individual mmf's.

As a practical illustration of this technique we shall consider its application to a simple, isolated, overhead, two-wire power line. Let the wires be small compared to their spacing and represented by points OO' in Fig. 11·4.

Mapping the Individual Fields. For each wire by itself, the magnetic flux lines of course are circles concentric with the wire, while the magnetic equipotential lines (always normal to the flux lines) are radii of these circles as shown, in part, in Fig. 11·3.

Let the equipotential lines be spaced by amount $\Delta\mathfrak{F}$, and the flux lines by amount $\Delta\phi$, to produce curvilinear squares of reasonable size as in Fig. 11·3. It is convenient to draw the first flux line of diameter equal to the spacing of wires. From this arbitrary reference line the other lines may be located by setting up a relation which permits computation of the radius of each larger and each smaller circle successively. This relation will now be developed.

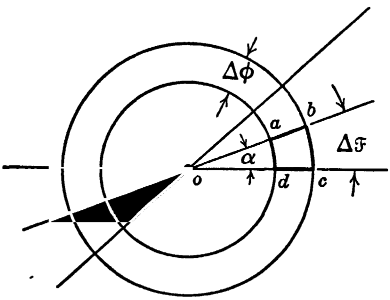

FIG. 11·3.

Computation for Forming the Squares. Consider the desired curvilinear square *abcd* in Fig. 11·3 where radius $R_1 = oa$ and $R_2 = ob > oa$.

$$\text{Arc } ad = 2\pi R_1 \frac{\alpha}{360} = \frac{\pi\alpha}{180} R_1$$

$$\text{Arc } bc = 2\pi R_2 \frac{\alpha}{360} = \frac{\pi\alpha}{180} R_2$$

For curvilinear squares it is required that

$$(ab + cd) = (ad + bc)$$

For the left side of this equation we observe that

$$ab = cd = R_2 - R_1$$

For the right side we observe that

$$ad + bc = \frac{\pi\alpha}{180} R_1 + \frac{\pi\alpha}{180} R_2 = \frac{\pi\alpha}{180} (R_1 + R_2)$$

We now substitute these to give

$$2(R_2 - R_1) = \frac{\pi\alpha}{180} (R_1 + R_2)$$

or

$$R_2 - R_1 = \frac{\pi\alpha}{360} (R_1 + R_2)$$

transforming

$$R_2\left(1 - \frac{\pi\alpha}{360}\right) = R_1\left(1 + \frac{\pi\alpha}{360}\right)$$

and

$$\frac{R_2}{R_1} = \frac{1 + \dfrac{\pi\alpha}{360}}{1 - \dfrac{\pi\alpha}{360}}$$

Knowing α, the ratio R_2/R_1 for the radii of successive circles is easily found.

Superposing the Equipotential Lines. When the map is complete for each individual wire the superposition of the equipotential lines proceeds as in Fig. 11·4.

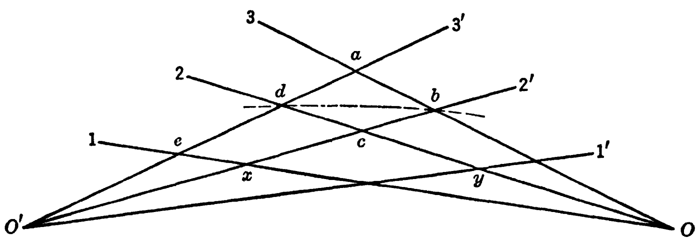

FIG. 11·4.

Consider points of intersection *abcd* of lines $O2$, $O3$, $O'2'$, and $O'3'$. If going from $O2$ to $O3$ (clockwise) represents a rise in potential $\Delta\mathfrak{F}$, it should be clear that for wire O' which has current in direction opposite to wire O, potential rise $\Delta\mathfrak{F}$ will be experienced in the counterclockwise direction from $O'2'$ to $O'3'$. From c to b is then a rise of amount $\Delta\mathfrak{F}$.

Because the interval between successive equipotential lines is made identical for both wires it follows that from c to d is the same rise as from c to b and that b and d are at the same potential.

A resultant equipotential line must pass through b and d, forming a curvilinear triangle bcd. By similar analysis of adjacent points of intersection the line is extended to closure. Other lines are started through ec, xy, etc., so long as the drawing permits.

Superposing the Flux Lines. The flux circles intersect as at $abcd$ in Fig. 11·5. The direction of flux lines due to wires O and O' with opposite currents is respectively clockwise and counterclockwise. Although the summation of these directional quantities is basically vectoral, it suffices for mapping merely to determine through which intersections of the *superposed* lines the *resultant* line should pass. In the region $abcd$ (Fig. 11·5) the directions of the intersecting circles indicate that the resultant line should pass from d to b and not in the reverse direction or in any manner between a and c. This is readily checked by observing that from d we arrive at b by following either the circle directions dcb or dab to the diagonally opposite intersection. It is impossible to go from a to c or vice versa by following the circle directions.

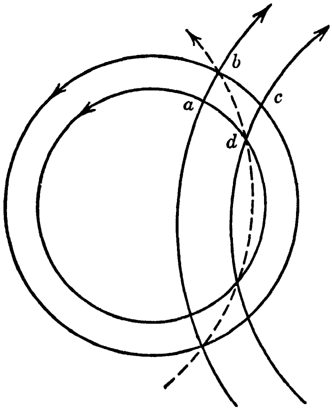

FIG. 11·5.

Continuation of this procedure, as for the equipotential lines, results in closure of the line. Other lines are similarly initiated and closed until the map is complete.

Shape of Resultant Field. It can be shown by appropriate mathematical procedure that both sets of resultant lines, flux and potential, are circles for this case. The flux lines are eccentric circles centered along a straight line through the axes of the wires. The potential lines all pass through the wire centers and are centered along a line normal to and bisecting the line joining the wire axes.

It is to be understood clearly that this is for wires of small diameter compared with their spacing. When the wires are not small enough to be considered as points the potential lines for each wire intersect at a point which does *not* coincide with the geometric center or axis of the wire.

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1. Eshbach, "Handbook of Engineering Fundamentals," John Wiley and Sons, pp. 8-18–8-27, 11-101–11-109.
2. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., Seventh Edition, 2-42, 2-46, 4-299–4-328.

QUESTIONS

11·1. (*a*) What is a gauss?

(*b*) What is an oersted?

11·2. For each of the following magnetic circuit quantities:

(*a*) Mmf.	(*c*) Permeance.	(*e*) Flux density.
(*b*) Flux.	(*d*) Potential gradient.	(*f*) Permeability.

Tabulate the following:

1. Letter symbol.
2. MKS unit of measure.
3. Analogous electric circuit quantity.

11·3. Expound the concept of specific permeability μ_s.

11·4. Expound the difference between rationalized and unrationalized systems of magnetic units.

11·5. Make a sketch of the B and H scales of Fig. 11·1 and add scales in the appropriate MKS units.

11·6. Explain mathematically how the diagonal lines in Fig. 11·2 are able to represent values of permeability.

11·7. From the fundamental relation between $\mathfrak{F}$ and ϕ show that, in a medium of constant permeability, magnetic potential gradient is proportional to density of flux.

11·8. (*a*) How is μ found from the B-H curve?

(*b*) Explain why permeability μ is *not* the slope of the B-H curve.

11·9. In mapping lines of magnetic flux and potential which extend into regions of different permeability show why the lines cannot form squares, curvilinear or otherwise, in both regions.

PROBLEMS

11·1. From the values of μ_0 given in the text compute its value for kiloline, ampere-turn, inch units.

11·2. Manufacturers of magnetic steels commonly present their data in CGS units. A typical silicon steel for transformer cores requires 2.1 ampere-turns per centimeter to produce 10 kilogauss, and 32 ampere-turns per centimeter to produce 15 kilogauss. For each pair of values compute:

(*a*) Equivalents in MKS units (rationalized, of course).

(*b*) Equivalents in kiloline, ampere-turn, inch units.

(*c*) Specific permeability μ_s.

11·3. A two-wire power line consists of No. 10 wires spaced 2 ft and carrying 24 amp. Map by the method of superposition described in this chapter the equipotential lines and flux lines, using scale 1 in. = 1 ft.

Use an 8 in. by $10\frac{1}{2}$ in. sheet of good quality unruled report paper and locate the two points representing the wire centers in the middle of the sheet.

Let $\Delta\mathfrak{F} = 1$ ampere-turn so that 24 radial equipotential lines are drawn for each wire.

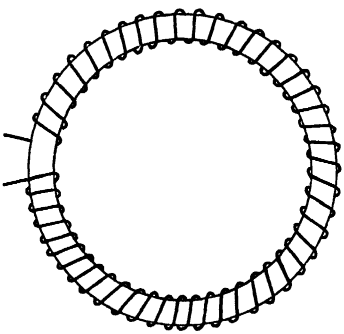
Draw as many flux circles for each conductor current as space permits and none smaller than about $\frac{3}{8}$ in. radius.

Accurate draftsmanship is essential to a successful final picture.

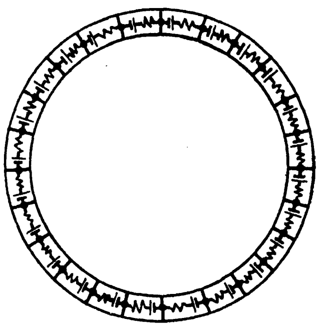
Chapter XII

MAGNETIC CIRCUITS

12·1. Computation of Magnetic Circuits—Toroidal Coil. Because (as before mentioned) there are no magnetic insulators, the computation of magnetic circuits in general is complicated by our inability to confine the magnetic field to the desired path. The only type of magnetic circuit which practically succeeds in confining the magnetic field to a finite path is what may be termed a distributed short-circuited circuit.



(a)



(b)

FIG. 12·1. Magnetic toroid and analogous electric circuit.

This is best typified by the common toroidal or doughnut-shaped coil which may be visualized as a long closely wound solenoid bent so that its originally straight-line axis forms a complete circle. This is shown schematically in Fig. 12·1*a*.

It is helpful to consider the analogous electric circuit of Fig. 12·1*b*. Here a number of flashlight cells are represented as forming a closed loop. Each cell comprises a source of emf and an internal resistance which, on short circuit, is a load element so that each cell is at once both source and load. Because the cells themselves form the circuit without other conductor, the circuit approaches an ideal distributed circuit; a circuit in which the emf and the load, instead of being lumped at the ends of a line, are uniformly distributed throughout the entire circuit. That each cell is in effect individually short-circuited is clear

from the writing of a Kirchhoff loop equation. For n cells, each with emf E and resistance R, we have $+(E - IR) + (E - IR)$ etc. $= 0$.
or

$$n(E - IR) = 0$$

or

$$E - IR = 0$$

Clearly the potential difference between the *terminals* of each cell is zero.

Now the important observation is that even though the ring of cells be immersed in a good conducting medium, such as acidulated water or even mercury, there is *no difference in potential* available to cause current *outside* of the cells.

Similarly for the magnetic circuit of Fig. 12·1*a*, there is no magnetic potential difference to produce magnetic flux *outside* of the toroidal ring, i.e., outside the "dough" of the doughnut. Each turn of the coil produces ampere-turns which are just sufficient to maintain the internal flux in the wafer-like slice of doughnut on which the coil turn is wound. Letting $\mathcal{F}$ = ampere-turns of one turn and $\mathcal{P}$ = permeance of the "wafer," $\phi = \mathcal{P}\mathcal{F}$ is the internal flux which makes the magnetic potential between the faces of the one-turn-on-a-wafer element zero.

There is this one point of difference: the progression from turn to turn inherent in the winding of the coil accumulates into one turn concentric with the hole in the doughnut so that some flux will thread the hole. When the toroidal core is of ferromagnetic material this is usually unimportant, but for an air core or the equivalent it is sometimes desirable to nullify the effect of this accumulated turn by bringing one end connection of the coil back along the coil axis so as to form one turn carrying current in direction opposite to that of the progression through the coil winding.

Toroidal coils are commonly used for some types of transformers and choke coils, especially where precise measurement or standards for measurement are desired.

12·2. In General—Assumptions Here. In the general case for ferromagnetic circuits the coils which provide the mmf sources are concentrated rather than distributed. The magnetic flux external to the intended magnetic circuit is referred to as ***leakage flux*** or stray flux. While this is commonly an important factor the computation which follows will assume that all the flux is confined to the indicated magnetic circuit. Furthermore we shall assume that the density of flux is uniform throughout any cross section and that there is no bulging or fringing of flux in small air gaps. These assumptions may produce considerable error in the result and it is commonly necessary to apply empirical or calculated corrections in practice.

12·3. Kirchhoff's Laws Applied to Magnetic Circuits. The analogy between the electric and magnetic circuit previously exploited carries through to include the application of Kirchhoff's laws. Because Kirchhoff's loop equation applies to potential differences in general (mechanical, thermal, etc.) it is but to be expected that it should apply to the magnetic potential differences of magnetic circuits. Thus we write $\Sigma pd = 0$ or $\Sigma \mathfrak{F} = 0$ and make our traverses quite as for the electric circuit.

Remembering that the lines used to represent magnetic fields are always continuous or closed lines with specific direction, it should be clear that for any junction in a magnetic circuit the algebraic summation of all flux lines to the junction must be zero, i.e., $\Sigma\phi = 0$ quite as $\Sigma i = 0$ for the electric circuit junction.

12·4. Polarity of Magnetic Potential. In order to apply Kirchhoff's Σpd to the magnetic circuit it is evident that some means must be available during the traverse for distinguishing potential rise from potential fall. Unfortunately, in the first introductory studies of magnetic phenomena it is usual to avoid entirely any recognition that magnetic polarity in the true sense of the term exists at all. "Oh yes," says the reader, "we know all about north and south poles; that is not new!" Quite so, but unfortunately *there is no correspondence between the polarity of magnetic potential and* N *and* S *poles!* N and S, with the exception of the terrestrial poles, merely designate places where *flux* respectively leaves and enters a portion of a magnetic circuit, as shown in Fig. 12·2*a*. The terrestrial magnetic poles, of course, are defined oppositely and are considered to be *the* N and S poles. All others, in strict academic parlance, are merely N-*seeking* and S-*seeking* poles.

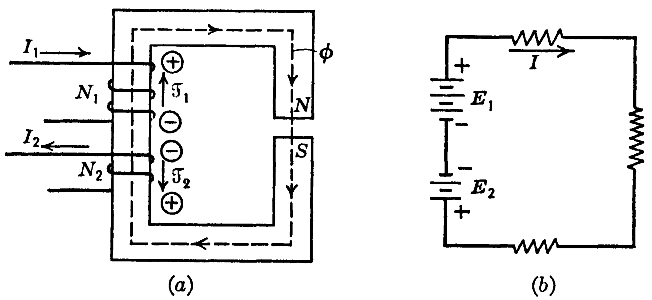


FIG. 12·2. A two-coil magnetic circuit and analogous electric circuit.

To insure that the true concept of magnetic polarity may be entirely clear, let us consider Fig. 12·2*a* which shows a simple ferromagnetic ring with two coils and Fig. 12·2*b* which shows the analogous electric circuit.

From the well-known right-hand or corkscrew rule it is clear that current I_1, flowing as shown in coil N_1, will produce flux ϕ in the clockwise direction shown. Likewise it is clear that I_2 flowing in the opposite direction through a similarly wound coil N_2 will, by itself, produce flux in the opposite direction. Now if both currents flow simultaneously and if I_1N_1 and I_2N_2 are such that $\mathfrak{F}_1 > \mathfrak{F}_2$ the resultant flux will have the direction determined by $\mathfrak{F}_1$, or clockwise as shown by ϕ. It can be seen that the actual flux direction through coil N_2 is no longer in accord with the corkscrew rule. The rule, as commonly stated, is then not really a fundamental one. This is because the corkscrew rule *actually* indicates *only the magnetic polarity* of the coil and not necessarily the direction of flux except when the coil is sole agent for producing the flux.

The issue is clear in the analogous electric circuit of Fig. 12·2*b*. If we were so rash as to make a statement that current always flows through a dry cell from zinc to carbon instead of stating that the polarity of the dry cell is zinc negative and carbon positive we would encounter the same difficulty with E_2 that has been cited for $\mathfrak{F}_2$.

It is imperative that some means for indicating magnetic polarity be employed. There is no such indicator, corresponding to the electric + and −, provided by standard symbolism. Two devices suggest themselves as follows.

1. The arrow direction erroneously associated with flux in the right-hand or corkscrew rule may be utilized to mean *direction of magnetic potential rise.*
2. Specially identified signs ⊕ and ⊖ may be used analogously as in the electric circuit.

Both of these are shown in Fig. 12·2*a*.

12·5. The Attack. The simplest quantitative magnetic circuit problems concern the computation of the mmf required to produce a desired flux in some part of a given magnetic circuit structure. The converse problem, and others including the computation of dimensions of parts, are as common as the analogous electric circuit problems but are sometimes relatively more difficult of attack. It is recommended that, at first, the student construct the analogous electric circuit for each problem and capitalize on his knowledge of electric circuits to guide him, especially in the application of Kirchhoff's laws.

Because in general it is necessary to employ *B-H* curves instead of fixed values of μ, the procedure differs appreciably from the all-algebraic attack commonly permissible in the electric circuit. In order to find

the mmf required to produce flux ϕ in some particular part of the magnetic circuit it is necessary to compute the flux density B, find from the B-H curve of the given material its corresponding magnetic potential gradient H, and then multiply H by the length of flux path in the given circuit part to determine the mmf. These Hl values correspond to the IR drops of the analogous electric circuit even though they are not computed as $\phi\mathcal{R}$ drops. The magnetic polarity of an Hl drop always constitutes a fall with the direction of flux, just as in the analogous electric circuit. When traversing the circuit *with* the flux direction, an Hl is taken to be algebraically *minus*. When traversing *against* the flux direction, the Hl, of course, is taken *plus*.

The details of the procedure are best illustrated by the solution of a few typical problems. Because the number of quantities involved in the solution of magnetic circuits is considerably greater than for the electric circuit it is highly advisable to use a tabulation of all quantities in the following manner. Observe clearly that this in no way takes the place of writing the Kirchhoff equations or other pertinent relations

Part	Material	ϕ	A	B	H	l	Hl	$\mathcal{F}_g$

among the tabulated values. Note that separate columns are provided for the Hl drops and for the coil or source mmf's $\mathcal{F}_g$. It is imperative that the magnetic circuit be *labeled* with sufficient letters to avoid ambiguity in the identification of the circuit *parts*.

For the special purpose of magnetic circuit computation the B-H curves, as explained in the preceding chapter, are here expressed in kilogauss and ampere-turn per centimeter units (Fig. 12·3*ab*) because the meter dimension of the straight MKS units is awkwardly large. Curves for the same materials are also provided here in the English system (Fig. 12·4) with kilolines per square inch and ampere-turns per inch for use with problems dimensioned in inches, and in accord with a considerable degree of industrial practice.

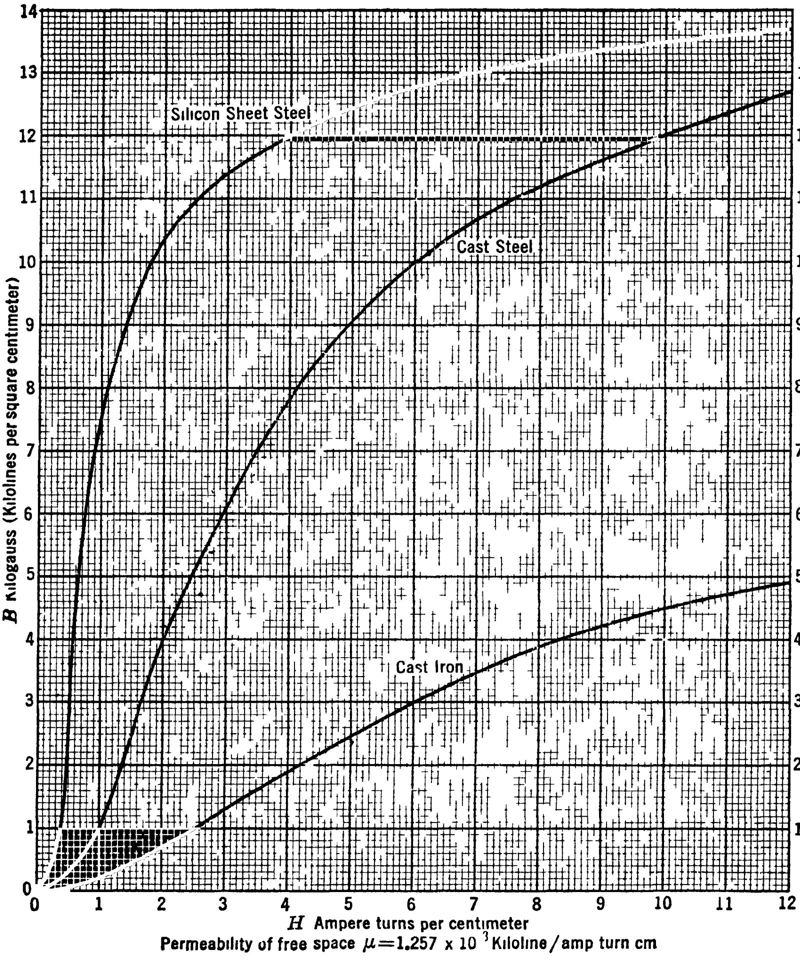

14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
Silicon Sheet Steel
Cast Steel
Cast Iron
B Kilogauss (Kilolines per square centimeter)
0 1 2 3 4 5 6 7 8 9 10 11 12
H Ampere turns per centimeter
Permeability of free space μ=1.257 x 10 3 Kiloline/amp turn cm

FIG 12 3a

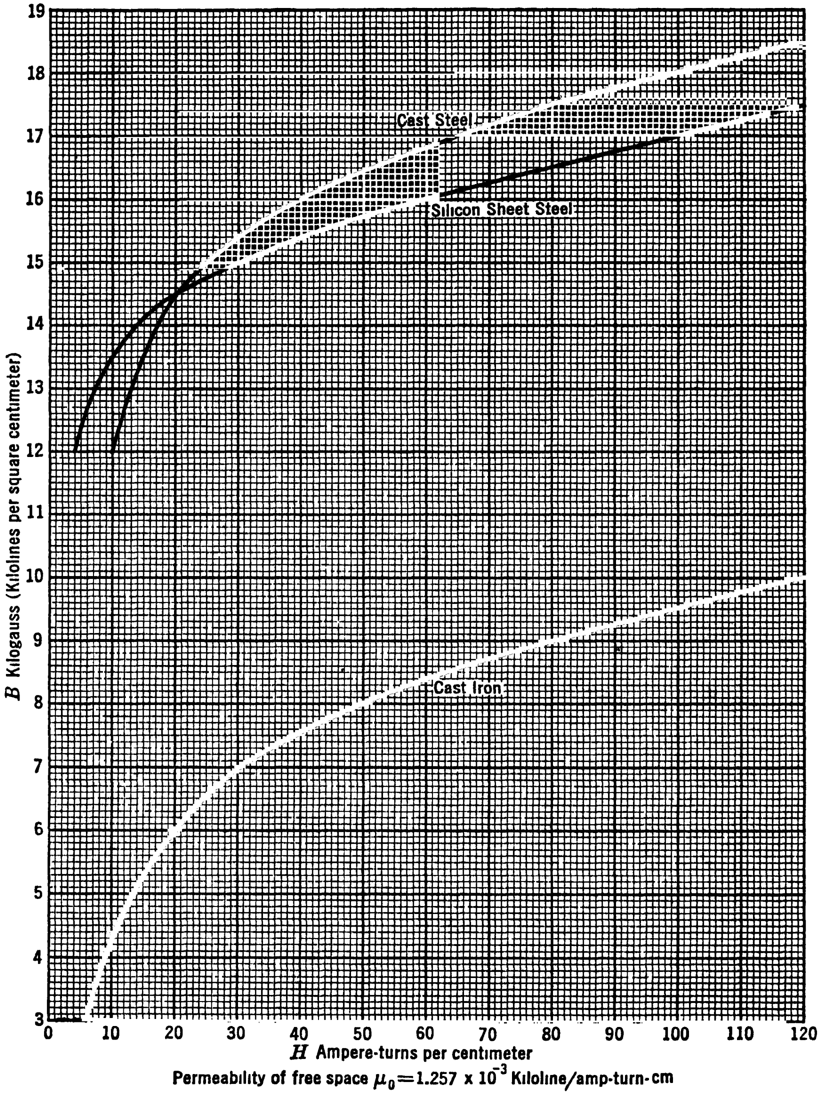

Cast Steel
Silicon Sheet Steel
Cast Iron
B Kilogauss (Kilolines per square centimeter)
H Ampere-turns per centimeter
Permeability of free space $\mu_0 = 1.257 \times 10^{-3}$ Kiloline/amp-turn-cm

FIG. 12·3b.

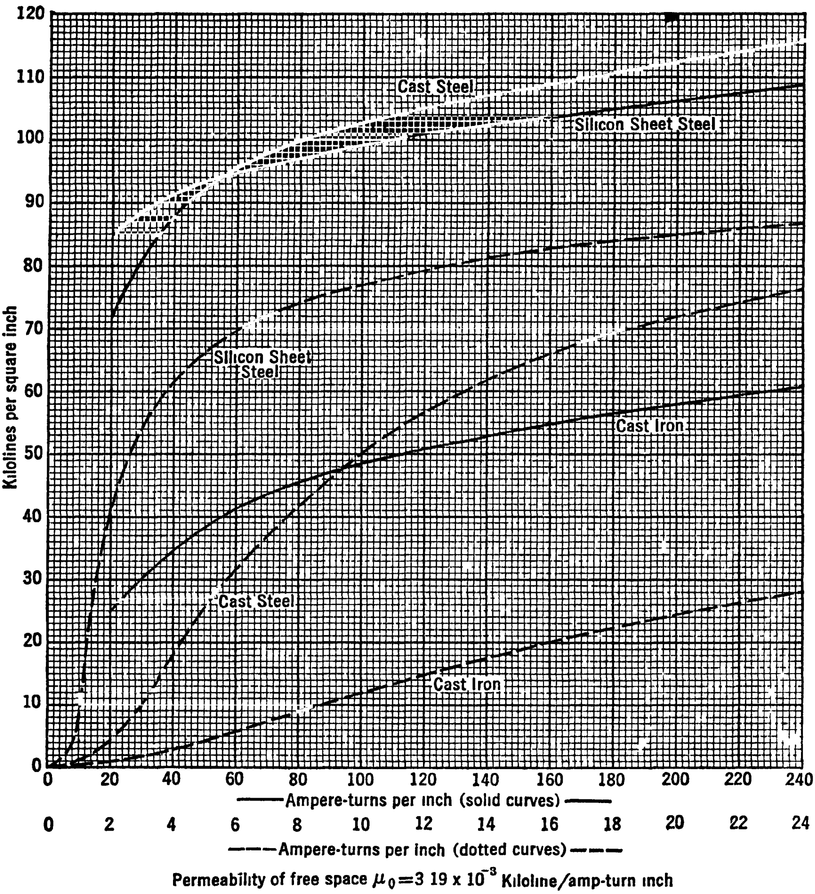


FIG. 12·4.

12·6. Series Circuit.

Example 1. Given the magnetic circuit of Fig. 12·5*a* find $\mathfrak{F}$. Recording data as in table on page 189, the sequence of computation is

$$B_{ac} = \frac{\phi}{A} = \frac{90}{15} = 6.0 \text{ kilogauss}$$

$$B_{abc} = B_{ac}$$

$$H_{ac} = \frac{B}{\mu_a} = \frac{6.0}{1.257 \times 10^{-3}} = 4770 \text{ ampere-turns per centimeter.}$$

Part	Material	ϕ Kilolines	A Sq cm	B Kilogauss	H Amp-turn/cm	l cm	Hl Amp-turn	$\mathfrak{F}_g$ Amp-turn
ac	Air	90	15			0.063		
abc	CI	90	15			40		

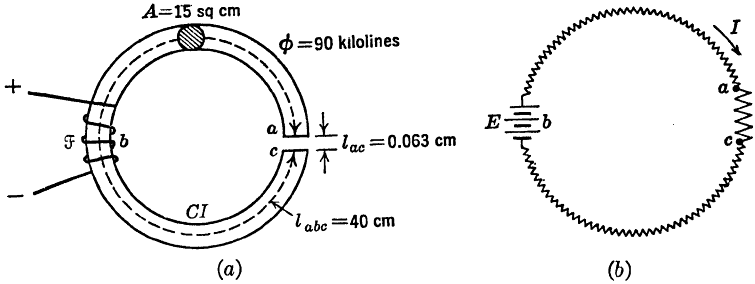


FIG. 12·5. A series magnetic circuit with electric circuit analog.

From B-H curve for CI (Fig. 12·3b), given $B_{abc} = 6.0$, we find $H_{abc} = 20$ ampere-turns per centimeter

$$(Hl)_{abc} = 20 \times 40 = 800 \text{ ampere-turns}$$

$$(Hl)_{ac} = 4770 \times 0.063 = 300$$

Applying $\Sigma\mathfrak{F} = 0$

$$(Hl)_{ac} + (Hl)_{abc} - \mathfrak{F}_b = 0$$

$$\mathfrak{F}_b = (Hl)_{ac} + (Hl)_{abc} = 300 + 800 = 1100 \text{ ampere-turns}$$

The table now appears as follows.

Part	Material	ϕ Kilolines	A Sq cm	B Kilogauss	H Amp-turn/cm	l cm	Hl Amp-turn	$\mathfrak{F}_g$ Amp-turn
ac	Air	90	15	6.0	4770	0.063	300	
abc	CI	90	15	6.0	20	40	800	1100

It is well perhaps to observe here that the actual magnetic potential difference, for that part of the magnetic circuit abc which contains the coil or mmf source, does not appear in the table. This value is not commonly needed directly, but sometimes puzzles the student who may

look for it. It is always found as the difference between the Hl and $\mathfrak{F}_g$ values for that *part*; in the above case $1100 - 800 = 300$. This value, of course, must check with the magnetic potential difference computed between a and c by any path. In this simple case $(Hl)_{ac} = 300$ provides the only check.

12·7. Series Circuit Requiring Guess-and-Test.

Example 2. The converse of this problem requires a guess-and-test method. Given (for the same circuit) the following data, to find ϕ_{ac}.

Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
ac	Air		15			0.063		
abc	CI		15			40		1100

Now, although it is known that $\mathfrak{F}_b = (Hl)_{ac} + (Hl)_{abc} = 1100$, neither Hl is directly determinate. It becomes necessary then to assume ϕ or B, compute $\mathfrak{F}_b$ for it, and repeat until the correct ϕ is found. Let us assume $B = 5$ kilogauss (there is but one B where A is uniform throughout the circuit). The table becomes

Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
ac	Air		15	5.0	3980	0.063	250	
abc	CI		15	5.0	13	40	520	770

Since $770 < 1100$, assume greater $B = 7.0$ and find

ac	Air		15	7.0	5570	0.063	350	
abc	CI		15	7.0	30	40	1200	1550

Since $1550 > 1100 > 770$, we have "bridged" the desired value. It is best now that these data be plotted roughly as in Fig. 12·6, *assuming* straight line for *short* length between points. From the plot an assumption $B = 6$ would probably be chosen and tested as before. Then

$$\phi = BA = 6 \times 15 = 90 \text{ kilolines} \qquad (Ans.)$$

This method is required, of course, in any series circuit comprising parts with different permeabilities when $\mathfrak{F}_g$ is given and ϕ (or B) is

unknown. *Both* $\mathfrak{F}_g$ and ϕ *may* be given, to *find* l_{ac} (for instance). Direct solution is then possible.

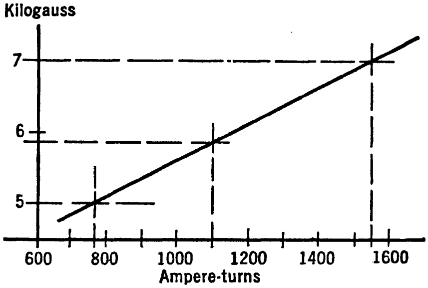


FIG. 12·6. Plot of values for guess-and-test solution.

12·8. Series-Parallel Circuits. While magnetic circuits rarely attain a complexity comparable with the network of the electric circuit, they are by no means confined to the simple series cases just considered. Ability to handle the series-parallel magnetic circuit easily suffices for all usual problems.

Because permeabilities are not constant for ferromagnetic materials, the use of an equivalent permeance method, corresponding (reciprocally) to the equivalent resistance method of the electric circuit, is definitely *not* permissible even when there is but one source of mmf. Kirchhoff's laws are indispensable for the solution of this kind of problem.

It should be clear that the principle of superposition in any form is *not* applicable to ferromagnetic materials except possibly as an approximate attack under very restricted conditions not to be considered or used in any of this study.

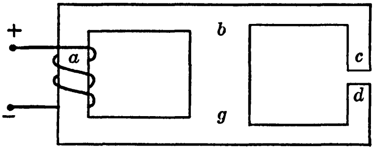
Let us first consider a circuit which has two parallel *load* branches and only one source of mmf.

Example 3. Given the circuit of Fig. 12·7 and the known values shown in the table below.

Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
bc	CS		4			6.0		
cd	Air		4	9.0		0.02		
dg	CS		4			8.0		
gb	CI		8			4.26		
bag	SS		6.25			15.0		

Solution. From the given polarity of the coil emf, direction of current and of fluxes can be found. Fluxes should be added to Fig. 12·7*a* with directions *gab*, *bg*, *bcdg*. Applying Kirchhoff's laws,

1. Loop *gab*: $+ \mathfrak{F}_a - (Hl)_{gab} - (Hl)_{bg} = 0$
2. Loop *bcdgb*: $- (Hl)_{bc} - (Hl)_{cd} - (Hl)_{dg} + (Hl)_{gb} = 0$
3. Junction *b*: $+ \phi_{ab} - \phi_{bg} - \phi_{bc} = 0$



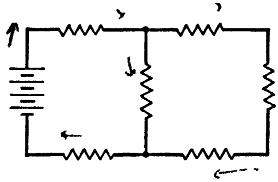


FIG. 12·7. A series-parallel magnetic circuit with electric circuit analog.

Now from the given $B_{cd} = 9$, we find $\phi_{cd} = BA = 9 \times 4 = 36$ kilolines. This is also ϕ_{bc} and ϕ_{dg}. We find $B_{bc} = B_{cd} = 9$ and then from *B-H* curve of CS find $H_{bc} = H_{dg}$. H_{cd} in air is computed as usual and the *Hl*'s tabulated.

It is now possible to use equation 2 to find

$$(Hl)_{gb} = (Hl)_{bc} + (Hl)_{cd} + (Hl)_{dg} = 30 + 143 + 40 = 213 \text{ ampere-turns}$$

Finding H_{gb} we get B_{gb} from the CI curve and compute ϕ_{gb} as tabulated.

From equation 3

$$\phi_{ab} = \phi_{bg} + \phi_{bc} = 64 + 36 = 100$$

After finding B_{ab}, H_{gab}, and $(Hl)_{gab}$, equation 1 discloses

$$\mathfrak{F}_a = (Hl)_{bag} + (Hl)_{gb} = 900 + 213 = 1113 \text{ ampere-turns} \qquad (Ans.)$$

The complete table is now:

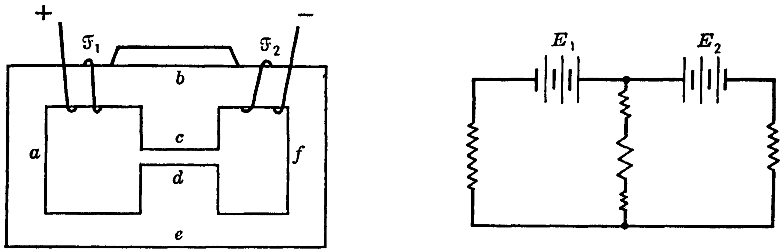
Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
bc	CS	36.0	4	9.0	5	6.0	30	
cd	Air	36.0	4	9.0	7150	0.02	143	
dg	CS	36.0	4	9.0	5	8.0	40	
gb	CI	64	8	8.0	50	4.26	213	
bag	SS	100	6.25	16	60	15.0	900	1113

Warning. It is to be clearly observed that, in general, the coil mmf $\mathfrak{F}_g$ is *not a bookkeeper's total* of the *Hl* column in the table. Because such

a total may be discovered to work for series magnetic circuits there is some temptation to assume that it is valid in general. To attempt to set up short cuts in lieu of the seemingly arduous application of Kirchhoff's laws is distinctly inviting trouble.

12·9. Magnetic Circuits with More Than One Source of Mmf. We shall now consider a series-parallel magnetic circuit with more than one source of mmf.

Example 4. Given the circuit and data as follows, to find $\mathfrak{F}_2$.

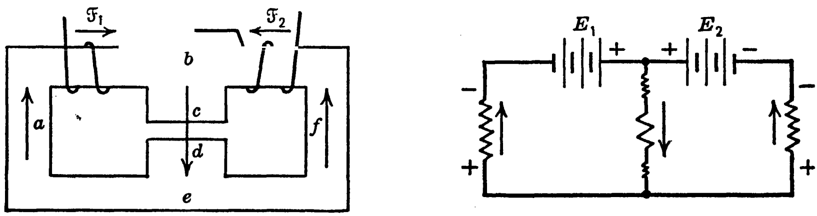


(a) (b)

FIG. 12·8. A two-source series-parallel magnetic circuit with electric circuit analog.

Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
bc	CI		10			7		
cd	Air	60	10			0.2		
de	CI		10			8		
bae	CS		2			30		654
bfe	CS		8.9			25		?

Polarities and directions are added as follows.



(a) (b)

FIG. 12·9.

By applying Kirchhoff's laws we find three equations.

1. $\phi_a - \phi_c + \phi_f = 0$
2. $(Hl)_{de} + (Hl)_{cd} + (Hl)_{bc} + (Hl)_a - \mathfrak{F}_1 = 0$
3. $(Hl)_{de} + (Hl)_{cd} + (Hl)_{bc} + (Hl)_f - \mathfrak{F}_2 = 0$

From 2 $$(Hl)_a = \mathfrak{F}_1 - (Hl)_{de} - (Hl)_{cd} - (Hl)_{bc}$$

From 1 $$\phi_f = \phi_c - \phi_a$$

From 3 $$\mathfrak{F}_2 = (Hl)_{de} + (Hl)_{cd} + (Hl)_{bc} + (Hl)_f$$

In conjunction with the table and B-H curves we compute the numerical values.

From 2 $$(Hl)_a = \mathfrak{F}_1 - (Hl)_{de} - (Hl)_{cd} - (Hl)_{bc}$$
$$= 654 - 160 - 954 - 140 = -600$$

From 1 $$\phi_f = \phi_c - \phi_a$$
$$= 60 - (-29) = 89$$

From 3 $$\mathfrak{F}_2 = (Hl)_{de} + (Hl)_{cd} + (Hl)_{bc} + (Hl)_f$$
$$= 160 + 954 + 140 + 150 = 1404$$

Part	Material	ϕ	A	B	H	l	Hl	$\mathfrak{F}_g$
bc	CI	60	10	6	20	7	140	
cd	Air	60	10	6	4770	0.2	954	
de	CI	60	10	6	20	8	160	
bae	CS	−29	2	−14.5	−20	30	−600	654
bfe	CS	89	8.9	10	6	25	150	1404

12·10. Permanent Magnets. Ferromagnetic materials all have in varying degree a property known as *magnetic hysteresis* which is manifested by a discrepancy between the values of flux for *de*creasing mmf as compared to those for the same *in*creasing values of mmf. This is commonly represented as in Fig. 12·10 by a B-H curve for a complete cycle of H taken from some maximum value a down through zero to the same maximum d with reversed polarity and back through zero to the starting point a. For $H = 0$ the discrepancy in B is be in Fig. 12·10. This curve is known as a ***hysteresis loop.*** The curve oa, known as the *normal B-H curve,* is obtained only when magnetization is begun with

the material in the completely demagnetized state and can proceed only in the direction o toward a.

For most purposes hysteresis is an undesirable property and much research in materials has been done to determine how it can be controlled. This has produced a group of steels, mostly silicon steels, for use in a-c equipment, and other apparatus with fluctuating fluxes, which assist the machine designer to keep the energy loss and distorting effects of hysteresis within acceptable limits.

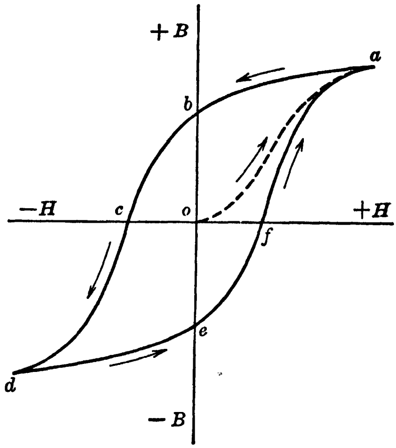


FIG. 12·10. A typical hysteresis loop for permanent magnet steel.

The portion of hysteresis loop known as the *demagnetizing curve* (bc in Fig. 12·10) extends from the value of B when $H = 0$, which is known as ***residual magnetism*** B_r, to the value of H when $B = 0$, which is known as ***coercive force*** H_c. It is this *retentive* characteristic of hysteretic materials which makes possible the so-called *permanent* magnet and has inspired considerable research also for materials with *more* hysteresis. Beginning with the long-established tungsten steels, the last decade has introduced cobalt steels and aluminum-nickel-cobalt (Al-Ni-Co) alloys known as Alnico which have greatly extended the applicability of such sources of magnetic flux and provided successful competition with electromagnets in many instances. They have also enabled improvements in the design and performance of long-established permanent magnet equipment such as measuring instruments (cf. D'Arsonval, Chap. III), magnetos, dynamic speakers, and magnetic separators, to suggest a few. While the Alnicos cost more per pound and must be shaped by casting and grinding instead of machining, they and other materials such as the still more costly iron-nickel-titanium-cobalt alloys known as the new K.S. magnet steels, alloys of copper-nickel-cobalt, alloys of the precious metals, and many others on the development horizon may be expected to attain an increasingly important place in magnetic equipment, which demands some consideration here of their place in magnetic circuit computations.

12·11. Computation of Permanent Magnet Circuits. The insertion of a section of permanent magnet (p-m) material in a magnetic circuit in place of a coil source of mmf necessarily introduces a different Hl drop

in this part of the circuit. Instead of attempting to analyze the performance of the p-m material into source mmf $\mathfrak{F}_g$ and Hl drop, it is customary to utilize the second quadrant of its hysteresis loop (bc in Fig. 12·10) to provide the *net* or "terminal" mmf available for the resulting flux. For a unit cube of the material this flux is B and the *net* mmf is merely the corresponding value of H. The sign of H is minus because the B-H graph is made in the sense of magnetic potential *drop* for the plus flux direction ($+B$); the minus H for plus B thus indicates an actual potential rise along the flux direction or a *source* in the same sense

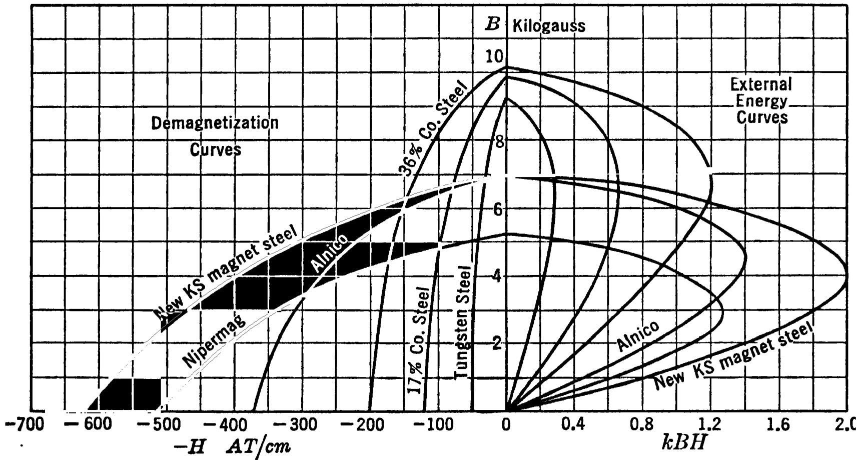


FIG. 12·11. Typical characteristics of permanent magnet steels.

as for an electric circuit. At point b the magnetic "generator" has $H = 0$ and $B = B_r$ which represents a magnetic *short-circuit* status. At point c the generator has $H = H_c$ and $B = 0$ which represents a magnetic *open-circuit* status. The short circuit occurs when the magnetic circuit is comprised entirely of the p-m material, while the open-circuit status is approached only as a limit when material which has a $+H$ (such as an air gap) is included in the circuit. Without magnetic insulators actual open circuit can only be simulated. Point c on the curve of course is obtained by sufficient reverse mmf from the coil source used to obtain the hysteresis loop and is obtained preferably with a toroidal magnetic circuit entirely of the p-m material (cf. Fig. 12·1a).

As shown in Fig. 12·11 for typical materials which have been magnetized well into the saturated region, it is the practice to plot B versus BH as well as the second quadrant B-H curve. The BH product gives the field energy (cf. Chap. XIII) which the unit cube of p-m material can establish in the magnetic circuit external to itself. For most eco-

nomic use of material it is customary to proportion the design so that the p-m material will function with the B-H values which provide the maximum external energy as shown by this B versus BH curve.

Let us observe the nature of the computation if we desire to substitute the Alnico of Fig. 12·11 in place of the coil source of $\mathfrak{F}$ in the magnetic circuit of example 1, page 188. If we use the same sectional area of Alnico as that of the CI it replaces, the flux density $B = 6$ kilogauss will enable the Alnico to provide $H = 140$ ampere-turns per centimeter. Since we are substituting this for the CI, which required $H = 20$ ampere-turns per centimeter, the equivalent H made available by the substitution must be $H = 140 + 20 = 160$, i.e., the net 140 provided by the Alnico plus the 20 relinquished by removal of the same length of CI. The length of Alnico is then

$$l = \frac{\mathfrak{F}}{H} = \frac{1100}{160} = 6.88 \text{ cm}$$

The volume of Alnico required is then

$$lA = 6.88 \times 15 = 103.2 \text{ cu cm}$$

Now let us compute the volume of Alnico required if we operate it at $B = 4.5$ kilogauss shown for the *maximum BH* value in Fig. 12·11. The sectional area must be

$$A = 15 \frac{6}{4.5} = 20.0 \text{ sq cm}$$

For $B = 4.5$ the Alnico $H = 250$ ampere-turns per centimeter. The equivalent replacement $H = 250 + 20 = 270$.

The length of Alnico is

$$l = \frac{1100}{270} = 4.08 \text{ cm}$$

and the volume is

$$lA = 4.08 \times 20.0 = 81.6 \text{ cu cm}$$

The saving by this design over the first is then

$$\frac{103.2 - 81.6}{103.2} = 20.9\%$$

The economic superiority of the second design is evident.

12·12. Leakage Flux. It must be clearly noted that, as for all computation in this chapter, leakage flux (flux not in the prescribed path) has been neglected. Although it is commonly not altogether negligible in electromagnet problems, it is a much more serious matter for the

p-m circuit computation. This may be attributed largely to the low margin of *B overload* which the p-m materials can take and still deliver *H*. For the above case the leakage flux may not only double but also possibly quadruple the density of flux computed in the Alnico. The solution just obtained would then be quite worthless. Experience and test data are necessary in most cases to determine the allowance for leakage fluxes.

12·13. Varying Flux. When permanent magnets are used as in magnetos, so that the flux is pulsating, the traverse of the hysteresis loop from *b* toward *c* (Fig. 12·10) will be periodically reversed. Any change along this reverse direction cannot follow the basic unidirectional hysteresis curve between *b* and *c* but will trace a small or *minor hysteresis loop* of its own somewhere within the area bounded by *bco*. This complication is not readily handled by any computation utilizing only the data provided by the basic hysteresis loop. Although graphical constructions for idealized characteristics are available *exact* results require additional test data.

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2. Roters, "Electromagnetic Devices," John Wiley and Sons, 1941, Chap. II and IV.
3. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., Seventh Edition, 4-393–4-412.
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QUESTIONS

12·1. Explain what is meant by an N pole and an S pole. Why have these poles no direct relation to magnetic polarity?

12·2. State and write symbolically the magnetic circuit laws corresponding to Kirchhoff's laws for the electric circuit.

12·3. Referring to the analog between electric and magnetic circuits, what in the latter corresponds to the internal resistance of a battery in the former? Explain fully.

12·4. Explain why a toroidal coil has practically no external magnetic field. Why "practically"?

12·5. Expound clearly the circumstances which make necessary a guess-and-test process in the computation of magnetic circuits.

12·6. Explain clearly why the well-known right-hand, or corkscrew, rule for relating direction of flux to direction of current through a coil applies only to a special case and what substitute is necessary when two or more coils are involved in a magnetic circuit.

12·7. Explain why the principle of superposition is not applicable to ferrous magnetic circuits.

12·8. Why, in a series magnetic circuit with more than one coil mmf, must a resultant of the mmf's of the coils be found instead of a resultant of the fluxes which the respective coil mmf's would produce, each by itself?

12·9. Make a sketch of the B and H scales on page 186 and add scales in the appropriate MKS units. Discuss the relative practicability of the two systems of units for this purpose.

12·10. Explain the meaning of *residual magnetism* and *coercive force.*

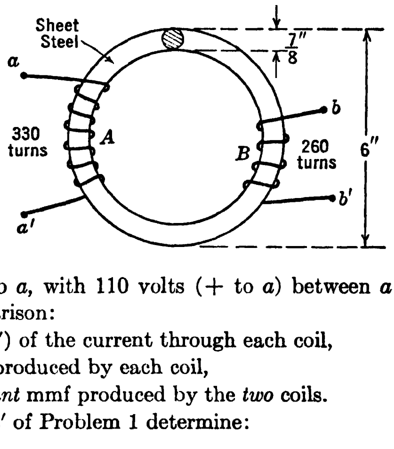
12·11. Explain why it is customary to use p-m materials each at a particular flux density.

12·12. In terms of B_r and H_c describe the advantage of Alnico over cobalt steel for permanent magnets. When either material may be suitable for a given application, how should their shapes (l versus A) compare for optimum use of material?

12·13. Why is it customary to shape Alnico magnets either by casting and grinding or by molding and sintering in powdered form?

PROBLEMS

12·1. The accompanying figure shows the arrangement of coils and core with dimensions used by Michael Faraday in the first successful experiment in electromagnetic induction on August 29, 1831, as recorded in his laboratory notes.



Coil aa' consists of 330 turns of wire having a resistance of 11 ohms and coil bb' has 260 turns with a resistance of 11 ohms.

For each of the following connections, namely,

1. b connected to a', and 220 volts (+ to a) between a and b',
2. b' connected to a', and 220 volts (+ to a) between a and b,
3. b connected to a, and b' to a', with 110 volts (+ to a) between a and a',
4. b connected to a', and b' connected to a, with 110 volts (+ to a) between a and a'. Determine and tabulate for comparison:

(*a*) Magnitude and direction (e.g., a to a') of the current through each coil,

(*b*) Magnitude and polarity of the mmf produced by each coil,

(*c*) Magnitude and polarity of the *resultant* mmf produced by the *two* coils.

12·2. If 10 amp flows through the *coil* aa' of Problem 1 determine:

(*a*) The flux density in the core.

(*b*) The total flux produced (neglect leakage).

(*c*) The flux density and the total flux if the cross-sectional area of the core were doubled. Assume mean path length unchanged.

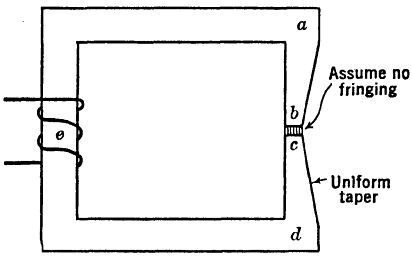
(*d*) The flux density and the total flux if the effective path length of the original ring were doubled. Assume original area of core.

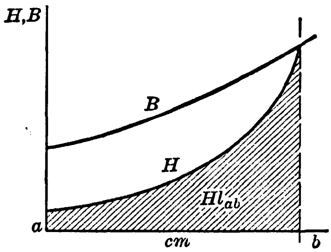
(*e*) The current required to produce *one-half* as much flux in the *original* ring.

(*f*) If the current is doubled, how wide an air gap may be cut through the core and yet maintain the same flux through the core? Assume length of iron remains unchanged.

12·3. The magnetic circuit in the figure is assembled from sheet steel laminations of ***uniform thickness.*** Use *B-H* curves of Fig. 12·3*ab*.

Part	Material	ϕ Kilolines	*A* Sq cm	*B* Kilogauss	*H* Amp-turn/cm	*l* cm	*Hl* Amp-turn	$\mathfrak{F}_g$ Amp-turn
bc	Air		5	14		0.01		
ab = *dc*	SS		10 to 5			10		
dea	SS		10			40		?





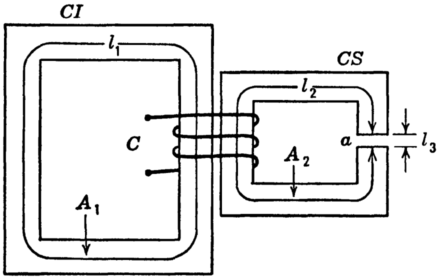
To find the magnetic potential *Hl* required for the tapered portion *ab*, plot on *graph paper* a curve of $B = f(l)$. From the *B-H* curve of sheet steel plot a curve of $H = f(l)$. Then the area under the latter curve will be the desired *Hl* for section *ab* as shown in the accompanying graph.

(*a*) Compute the coil mmf and find the current required for 100 turns.

(*b*) If the air gap length is doubled what current will be required for the same density of field in the gap?

(*c*) If, with the original air gap in *a*, the coil current is doubled, what density of air gap flux will result? (Guess and test required.)

12·4. Given, for the accompanying magnetic circuit:



l_1 = 68 cm (cast iron)
l_2 = 40 cm (cast steel)
l_3 = 0.2 cm (air gap)
A_1 = 5 sq cm
$A_2 = A_3$ = 4 sq cm

The flux through the air gap at *a* is to be 60 kilolines. Determine:

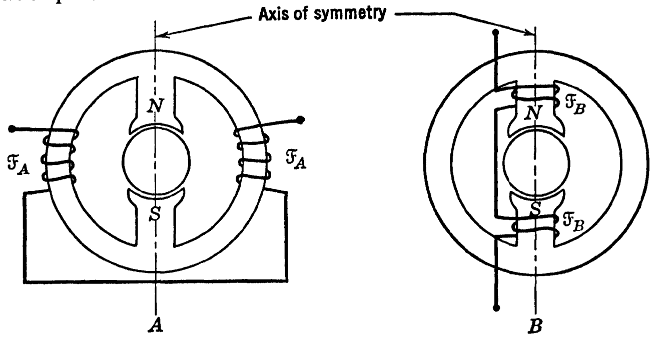
(*a*) The ampere-turns necessary for the coil *C*.

(*b*) The flux produced in the cast-iron ring.

(*c*) The total flux linking with the coil *C* (neglecting magnetic leakage).

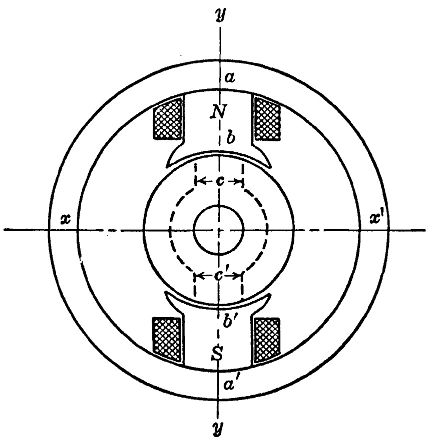
12·5. The magnetic circuits below represent an improvement made in the location of the coils on a generator in the early days of their development. While it is true that design *B* reduces the leakage flux, there is a more drastic improvement of funda-

mental character. Write the Kirchhoff equations for each case and study the coil mmf's required for a given flux, neglecting leakage. Compare $\mathfrak{F}_A$ with $\mathfrak{F}_B$ and explain the improvement.



12·6. The data in the table and sectional sketch apply to a two-pole d-c dynamo. The sectional areas and lengths are mean values and neglect all leakage fluxes. Note that but one-half of the structure about axis of symmetry *yy* need be considered.

Identity	Path	Material	ϕ Kilolines	A Sq cm	l cm
Pole	$ab = a'b'$	SS		8	10
Air gap	$bc = b'c'$	Air	120	16	0.25
Armature	$c\,c'$ (total)	SS		10	20
Yoke	$axa' = ax'a'$	CS		?	60



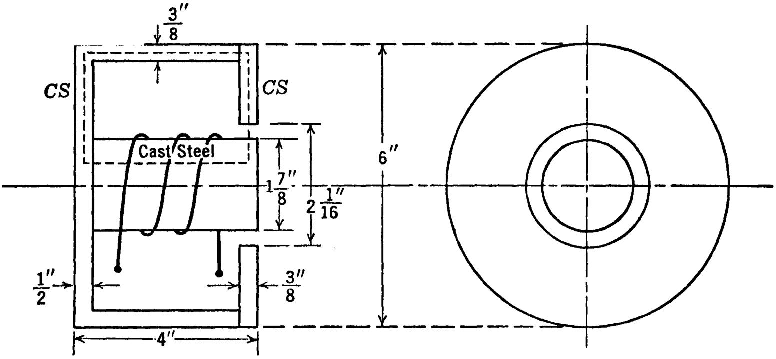
Compute:

(*a*) Cross-section of yoke which will make the yoke flux density ⅔ that in the poles.

(*b*) Ampere-turns required for each pole.

(*c*) Amperes required if each field coil has 1000 turns.

12·7. An auditorium type 15-in. dynamic speaker has a magnet structure as per sketch.



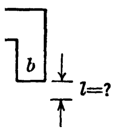
(*a*) How many coil turns are required for an air gap density of 110 kilolines per square inch and current 50 ma?

Note: Assume the length of the average flux path as shown by the dotted line. In the disk-shaped ends, where the flux density decreases toward the periphery, an acceptable approximation for the Hl's may be obtained by dividing the flux path through the disk into two equal lengths and using an average flux density for each.

(*b*) By what per cent would the required mmf be reduced if the air gap could be reduced to ½ the original specification with no change in the specification of air gap flux?

(*c*) If in (*b*) the original mmf were used with the reduced air gap what density of air gap flux would result?

12·8. The magnetic circuit shown here has the dimensions and materials given in the table. The total flux linking with the exciting winding is 100 kilolines.



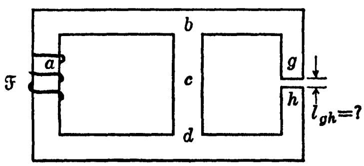
Path	Material	Length cm	Sectional Area sq cm
ab = *cd*	SS	20	3
acd	SS	40	4
dfa	SS	20	6.25
bc	Air	?	3

(*a*) What should be the length of the air gap *bc* in order that 40 per cent of the total flux may cross the air gap?

(*b*) How many turns are necessary on the exciting winding if the current is 10 amp?

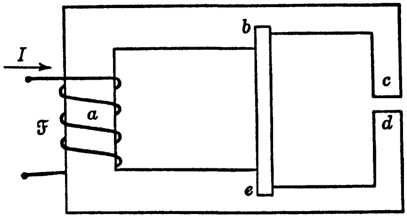
12·9. Total ampere-turns = 3440. Total flux through the exciting coil is 32 kilolines. Determine the length of the air gap *gh*.

Path	Mat'l	Length	Area
dab	CI	40 cm	4 sq cm
bg = *hd*	CI	15 "	4 " "
bcd	CI	12 "	2 " "
gh	Air	? "	4 " "

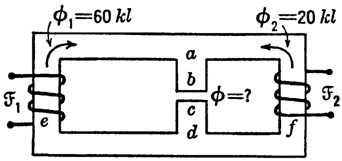


12·10. Indicate the paths and direction of the flux lines and determine:
(*a*) The total flux linking with the coil.
(*b*) The mmf to be supplied by the coil.

Part	Material	l cm	A sq cm	ϕ kl
cd	Air	0.1	6	90
bc = *de*	Sheet steel	20	6	
be	Cast iron	30	3	
eab	Sheet steel	50	11.7	



12·11. The flux linking with the first coil is $\phi_1 = 60$ kilolines, and with the second coil, $\phi_2 = 20$ kilolines, in the direction shown in the diagram. Indicate the paths of the flux lines, and determine:
(*a*) Flux ϕ which crosses the air gap.
(*b*) Magnitude and polarity of the mmf supplied by each respective coil.

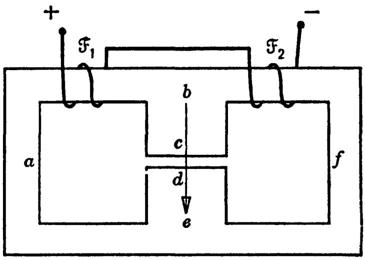


Part	Material	Length cm	Sectional Area sq cm
bc	Air gap	0.25	10
ab = *cd*	Cast iron	10	10
dea	Cast iron	60	10
dfa	Cast iron	80	10

12·12. Solve the preceding problem for the case when the flux ($\phi_2 = 20$ kilolines) is in the opposite direction to that shown in the figure.

12·13. Given the accompanying magnetic circuit and data compute the ampere-turns for coil $\mathfrak{F}_2$.

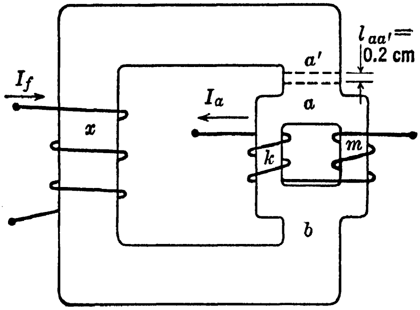
Part	Mat'l	ϕ kl	A sq cm	l cm	$\mathfrak{F}$ Amp-turn
bc	CS		6	6	
cd	Air	42	6	0.01	
de	CS		6	8	
bae	CI		12	25	605
bfe	CI		6	20	



12·14. Solve the preceding problem with the accompanying data.

Part	Mat'l	ϕ kl	A sq in	l in	$\mathfrak{F}$ Amp-turn
bc	CS		6	2	
cd	Air	90	6	0.005	
de	CS		6	3	
bae	CI		4	10	780
bfe	CI		3	8	

12·15. The following magnetic circuit represents a condition encountered in d-c dynamos where I_f is the field current and I_a is the armature current. By means of sufficient air gap aa' between rotor and stator the effect of I_a on the flux, which it is desired that only I_f determine, is reduced.



Part	Material	A sq cm	l cm
axb	Sheet steel	40	200
akb	" "	20	50
amb	" "	20	50

$$N_x = 5{,}000 \text{ turns}$$
$$N_k = N_m = 50 \text{ turns}$$

(*a*) When $I_f = 0$ and $I_a = 10.0$, determine ϕ_k, ϕ_m, and the magnetic potential $\mathfrak{F}_{ab}$ across ab. Prove $\phi_x = 0$.

(*b*) When $I_f = 0.20$ and $I_a = 0$, determine ϕ_x, ϕ_k, ϕ_m, $\mathfrak{F}_{ab}$.

(*c*) When $I_a = 10$, what I_f is required to give the same ϕ_x as in (*b*)?

(*d*) Note the values of $\mathfrak{F}_{ab}$ in the above and explain why I_a affects $\mathfrak{F}_{ab}$ when I_f is present (refer to *b* and *c*) although (*a*) demonstrates that I_a alone produces no $\mathfrak{F}_{ab}$.

(*e*) Compute $Hl_{aa'}$ for ϕ_x in the air gap and determine new values for I_f in (*b*) and (*c*) required to maintain the same ϕ_x.

(*f*) Compute the per cent change in I_f required to maintain ϕ_x when I_a changes from 0 to 10 for the case (1) no air gap, (2) with air gap.

(*g*) What effect has the air gap on the change in I_f occasioned by the presence of I_a, in this problem?

12·16. Compute the optimum area and length of p-m material required in place of each coil for the dynamo of Problem 6, using:

(*a*) Alnico.

(*b*) 36 per cent cobalt steel.

(*c*) Discuss the feasibility of each of these substitutions.

12·17. Compute the optimum area and length of p-m material required in place of the coil for Problem 8, using:

(*a*) Alnico.

(*b*) 36 per cent cobalt steel.

(*c*) Discuss the relative suitability of these materials for this particular case.

Chapter XIII

MAGNETIC FORCE—ELECTROMAGNETS

13·1. Magnetic Energy. In mechanics, work (energy of displacement) is computed by $W = \int F\,ds$. In hydraulics, $W = \int p\,dq$ where p = pressure encountered in the displacement of a quantity of liquid dq. In electric circuits, $W = \int e\,dq$ where e = emf encountered in the displacement of a quantity of electrons dq. Analogously in magnetic circuits, $W = \int \mathfrak{F}\,d\phi$ where $\mathfrak{F}$ = magnetic potential in which the flux $d\phi$ is created (or destroyed) and W is the energy stored (or released) in the field. This relation was introduced with magnetic potential in Chapter XI.

When a magnetic circuit is in air the permeance $\mathcal{P}$ is constant so that using $\mathfrak{F} = \phi/\mathcal{P}$ we can perform the integration required to find the field energy as follows.

$$W = \int \mathfrak{F}\,d\phi = \int^{\phi} \frac{\phi}{\mathcal{P}}\,d\phi = \frac{1}{2}\frac{\phi^2}{\mathcal{P}} \qquad [13\cdot1]$$

This may be expressed in either of two other forms.

$$W = \frac{1}{2}\frac{(\mathcal{P}\mathfrak{F})^2}{\mathcal{P}} = \frac{1}{2}\mathcal{P}\mathfrak{F}^2 \qquad [13\cdot2]$$

$$W = \frac{1}{2}\frac{\phi(\mathcal{P}\mathfrak{F})}{\mathcal{P}} = \frac{1}{2}\phi\mathfrak{F} \qquad [13\cdot3]$$

If we expand the last relation we find

$$W = \tfrac{1}{2}(BA)(Hl) = \tfrac{1}{2}BH(Al)$$

Since product Al represents the volume of the field it becomes evident that the field energy can be expressed as an energy density or energy per unit volume.

$$w = \frac{W}{Al} = \frac{1}{2}BH \qquad [13\cdot4]$$

or

$$w = \tfrac{1}{2}(H\mu_0)H = \tfrac{1}{2}H^2\mu_0 \qquad [13\cdot5]$$

or

$$w = \frac{1}{2}B\left(\frac{B}{\mu_0}\right) = \frac{B^2}{2\mu_0} \qquad [13\cdot6]$$

13·2. Magnetic Force. A very useful attack for finding mechanical forces in a system conceives the infinitesimal displacement ds of some part under such conditions that any energy change dW in the system is restricted to one resulting from mechanical work done on (or by) the system by virtue of the displacement. If the energy change in the system can be computed, it may be equated to the mechanical work, which is given by $dW = F\,ds$, and the force will then be: $F = dW/ds$. This is known as the ***principle of virtual displacement.***

For application to the magnetic case let us first consider the uniform magnetic field of a simple air gap as represented in Fig. 13·1. The field

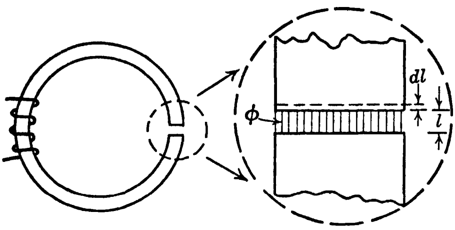


FIG. 13·1.

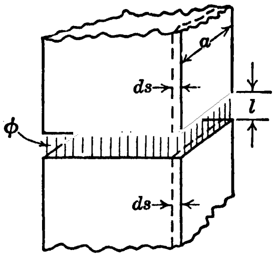


FIG. 13·2.

energy is expressed by equation 13·1, which may be expanded by substituting $\mathcal{P} = \mu \frac{A}{l}$ to give

$$W = \frac{1}{2}\phi^2 \frac{1}{\mathcal{P}} = \frac{1}{2}\phi^2 \frac{l}{\mu_0 A} = \frac{\phi^2}{2\mu_0 A}\, l \qquad [13\cdot7]$$

If now the air gap is increased amount dl, keeping ϕ constant, the field energy is increased by amount

$$dW = \frac{\phi^2}{2\mu_0 A}\, dl$$

With flux ϕ constant no emf can be induced in the energizing electric circuit and none of the field energy change can be accounted for by transfer to or from the electric circuit. Thus, the energy change must all be mechanical work $dW = F\,dl$, and there is a force

$$F = \frac{dW}{dl} = \frac{\phi^2}{2\mu_0 A} \text{ newtons (MKS units)} \qquad [13\cdot8]$$

This force must be one of *tension* in the gap because work had to be done on the system to increase the field energy during increase of the air gap.

The force may be expressed in terms of per unit area or *pressure* by the following treatment.

$$F = \frac{\phi^2}{2\mu_0 A} = \frac{(BA)^2}{2\mu_0 A} = \frac{B^2 A}{2\mu_0} \qquad [13\cdot 9]$$

$$\text{Pressure} = \frac{F}{A} = \frac{B^2}{2\mu_0} \qquad [13\cdot 10]$$

Now returning to equation 13·6 it becomes evident that ***the force per unit area** is simply equal to **the density of field energy.*** Two other forms of this relation corresponding to equations 13·4 and 13·5 are useful according to the specific problem.

13·3. Electromagnet Pull. Equation 13·8 is readily applied to ex- express the force of tension or pull utilized in many electromagnetic devices. Among these may be mentioned lifting magnets, automatic circuit breakers and motor controllers, relays, and telephone receivers or ear phones.

For a working formula we substitute 0.2248 lb = 1 newton and $\mu_0 = 4\pi \times 10^{-7}$ in equation 13·8 and obtain

$$F = \frac{\phi^2 \times 0.2248}{2 \times 4\pi \times 10^{-7} A} = 8.94 \times 10^4 \frac{\phi^2}{A} \text{ lb.} \qquad [13\cdot 11]$$

where ϕ is in webers and A in square meters.

For ϕ in kilolines and A in square inches we find

$$F = 8.94 \times 10^4 \frac{(\phi \times 10^{-5})^2}{(2.54 \times 10^{-2})^2 A} = 0.0139 \frac{\phi^2}{A} \text{ lb.} \qquad [13\cdot 12]$$

13·4. Force versus Length of Air Gap. The above relations, such as 13·12, apparently ignore the length l of the air gap. Some effort is required to reconcile this finding with the commonplace experience that more gap results in less force or pull for an electromagnet. There are two reasons for this reduction of pull.

1. Increase in gap length l reduces $\mathcal{P}$ which for the usual fixed mmf $\mathfrak{F}$ means a reduction in flux $\phi = \mathcal{P}\mathfrak{F}$.
2. Increase in gap length l is accompanied by increased *fringing* of the flux, i.e., an increased section A.

Turning again to $F = \phi^2/(2\mu_0 A)$ it is apparent that force F is reduced by reduction of ϕ and (or) increase of A, the former being especially effective. If required, an approximate *formula* for the pull may be created to include the gap length l but it will *not* be a *fundamental* relation.

13·5. Transverse Magnetic Force. Magnetic fields produce other forces than the simple *longitudinal* tension or pull which tends to close an air gap. In Fig. 13·1 if the displacement had been executed as ds in Fig. 13·2, so as to reduce the area of the air gap by amount dA, we would still find a force.

Returning to equation 13·7 we find that a change in field energy with respect to area gives, for constant ϕ and l:

$$\frac{dW}{dA} = \frac{\phi^2 l}{2\mu_0}\left(-\frac{1}{A^2}\right)$$

Since *reduction* in area is expressed by $dA = -a\,ds$ (Fig. 13·2) we find the force to be

$$F = \frac{dW}{ds} = -a\frac{dW}{dA} = \frac{a\phi^2 l}{2\mu_0 A^2}$$

The pressure or force per unit *area of application* (not area A) in the gap is

$$\text{Pressure} = \frac{F}{al} = \frac{\phi^2}{2\mu_0 A^2} = \frac{B^2}{2\mu_0} \qquad [13\cdot 13]$$

Now it is to be observed that this *transverse* pressure has precisely the same value as the *longitudinal* pressure found in equation 13·10. As shown by equation 13·7, *reduction* of area A produces *increase* in field energy; therefore work is done *on* the system and the transverse force is one of *compression* in contrast to the longitudinal tension.

It is often helpful to visualize these forces by conceiving the flux lines to be elastic strands (rubber bands) in longitudinal tension and transverse compression. More scientifically, it is to be noted that the forces tend to reduce l and increase A which appear in the permeance relation $\mathcal{P} = \mu(A/l)$.

Both forces thus tend to increase the permeance so that we may state that ***the mechanical forces produced by magnetic fields are in such direction as will tend to change the conformation of a magnetic circuit into one of higher permeance.***

13·6. Applications of Transverse Force. Since the transverse force is one of *compression*, it is concerned with forces of *repulsion* rather than forces of attraction. Such would be true in Fig. 13·3 where like poles are adjacent. More commonly these forces occur in adjacent conductors which carry current in *opposite* directions as in a transformer. These forces are of particular concern under abnormal conditions of short circuit or lightning disturbance when, in spite of seemingly adequate mechanical design, the coils may be torn out of place. The

jumping ring apparatus of Fig. 13·4, so commonly exhibited in electrical displays, is of this class.

Many of these problems which involve current-carrying conductors as in Fig. 13·4 are more readily solved with the concept which we early expressed by $F = Bli$.

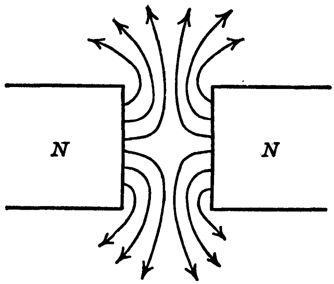


FIG. 13·3.

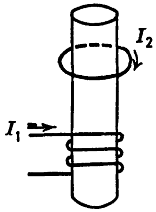


FIG. 13·4.

13·7. The Force between Parallel Wires. The $F = Bli$ relation is so closely associated with the transverse magnetic force relation, which we have just developed, that at times either one may be used. To illustrate this fact the two-wire transmission line, for which we studied the magnetic field by superposition in Chapter XI, serves admirably. Let the line be of considerable length and the wire diameter *small* as compared with the wire *spacing*. Let the conductors be spaced distance s

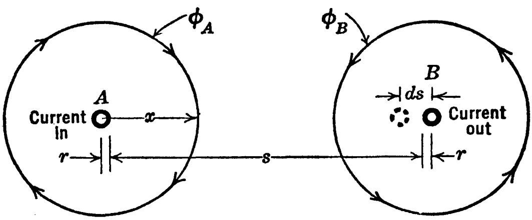


FIG. 13·5.

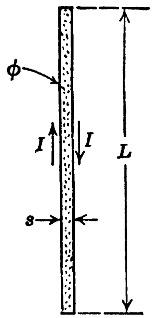


FIG. 13·6.

(Fig. 13·5) with A carrying current I *in* (away from the reader) while B carries return current I *out* (toward the reader).

The magnetic field ϕ which links with the elongated one-turn "coil" formed by the line (Fig. 13·6) represents energy $W = \dfrac{\phi^2}{2\mathcal{P}}$ (equation 13·1). If we move one conductor toward the other by amount ds

(Fig. 13·5), the permeance $\mathcal{P}$ of the linking path will be reduced. If, furthermore, we control current I so as to maintain the linking flux ϕ *constant,* no emf will be induced in the circuit and the increase in field energy $W = \dfrac{\phi^2}{2\mathcal{P}}$ must come *entirely from the mechanical work* done in moving the conductor. Thus we can find the force of repulsion between the two conductors by the *principle of virtual displacement.*

We must first find the flux ϕ linking the line loop. Fortunately, in a medium of constant permeability μ_0, this can be done by superposition of the fluxes ϕ_A and ϕ_B produced by the respective conductors A and B. Because these flux lines are circles concentric with each conductor (Fig. 13·5), the magnetic potential gradient is uniform around the path of each flux line and is easily computed in terms of the mmf $\mathcal{F} = NI = I$ and the length l of any flux line of radius x.

$$H_A = \frac{\mathcal{F}}{l} = \frac{I}{2\pi x} \qquad [13\cdot14]$$

The density of flux is then expressed by

$$B_A = H_A\mu_0 = \frac{I\mu_0}{2\pi x} \qquad [13\cdot15]$$

The flux between conductors due to A of radius r and length L is then

$$\phi_A = \int B_A dA = \int_r^{s+r} \frac{I\mu_0}{2\pi x} L\, dx$$

$$= \frac{I\mu_0 L}{2\pi}\int_r^{s+r}\frac{dx}{x} = \frac{I\mu_0 L}{2\pi}\ln\left(\frac{s+r}{r}\right) \qquad [13\cdot16]$$

The total linking flux due and external to both conductors becomes

$$\phi = \phi_A + \phi_B = 2\phi_A = \frac{I\mu_0 L}{\pi}\ln\left(\frac{s+r}{r}\right) \qquad [13\cdot17]$$

The permeance of this flux path is

$$\mathcal{P} = \frac{\phi}{\mathcal{F}} = \frac{\phi}{I} = \frac{\mu_0 L}{\pi}\ln\left(\frac{s+r}{r}\right) \qquad [13\cdot18]$$

And the field energy may be expressed (from equation 13·1) by

$$W = \frac{\phi^2}{2\mathcal{P}} = \frac{\phi^2\pi}{2\mu_0 L}\,\frac{1}{\ln\left(\dfrac{s+r}{r}\right)} \qquad [13\cdot19]$$

Now, *keeping ϕ constant*, the change in energy $(-dW)$ for displacement ds is

$$F = -\frac{dW}{ds} = \frac{\phi^2\pi}{2\mu_0 L} \frac{\left[\ln\left(\frac{s+r}{r}\right)\right]^{-2}}{s+r}$$

Putting back the value for ϕ we finally obtain

$$F = \left[\frac{I\mu_0 L}{\pi}\ln\left(\frac{s+r}{r}\right)\right]^2 \frac{\pi\left[\ln\left(\frac{s+r}{r}\right)\right]^{-2}}{2\mu_0 L(s+r)} = \frac{I^2\mu_0 L}{2\pi(s+r)} \qquad [13\cdot 20]$$

Since the radius r of each conductor is taken to be negligibly small as compared to the spacing s, equation 13·20 reduces to

$$F = \frac{I^2\mu_0 L}{2\pi s} \qquad [13\cdot 21]$$

This is done to avoid the considerable complication which would accompany consideration of the flux produced *within* the conductors. Radius r could not be assumed zero in the beginning without encountering difficulty with integration of flux in equation 13·16.

Now let us apply the alternative $F = Bli$ to this same problem. For this purpose we conceive that one wire, B, constitutes the current-carrying conductor which is located in the magnetic field produced by the other wire, A. We have already computed density of flux B_A produced by A at any distance x from the center of wire A. At conductor B, which is at distance s from conductor A (assuming negligible radii),

$$B_A = \frac{I\mu_0}{2\pi s} \qquad [13\cdot 22]$$

Substituting this value we readily obtain

$$F = \frac{I\mu_0}{2\pi s} LI = \frac{I^2\mu_0 L}{2\pi s} \qquad [13\cdot 23]$$

Thus we find that, depending on the physical structure involved in a particular case, the two force relations are more or less alternative although by no means completely so. It is not possible here to dwell further on the relative merit or applicability of these two concepts for detailed cases but the student will find occasion for further application of them.

It may be observed that by substituting the value of μ_0 in the above relation we have, in MKS units:

$$F = \frac{I^2(4\pi \times 10^{-7})L}{2\pi s} = 2I^2 \frac{L}{s} \times 10^{-7} \text{ newtons} \qquad [13 \cdot 24]$$

While this force is unlikely to be of consequence for transmission lines or ordinary distribution circuits it is important in the heavy current-carrying distribution buses in power plants, steel mills, electrolytic refineries, etc. The computation of such forces in power transformers and other motionless apparatus is but little less important than it is for motors and generators in successful modern electrical machine design. It should not be overlooked that whenever the field is of *uniform density* the simple relation of equation 13·13 may be applied directly.

13·8. Electromagnet Windings—Wire Size. In order to obtain the necessary ampere-turns $\mathfrak{F}$ from a given source of emf E to operate a magnetic circuit, say an electromagnet, some computation is necessary which will indicate the size of wire A and number of turns N to be used.

The basic relations are well known. For R the resistance of the entire winding or coil, and I the current through turns N to produce mmf $\mathfrak{F} = NI$, we apply the laws of Ohm and Davy to give

$$E = RI$$

where $R = \rho(l_mN/A)$ and l_m = mean length of one turn of wire of section A and resistivity ρ in the coil. Substituting, we find

$$E = \rho \frac{l_m N}{A} I = \rho \frac{l_m(NI)}{A} = \rho \frac{l_m \mathfrak{F}}{A} \qquad [13 \cdot 25]$$

$$A = \rho \frac{l_m \mathfrak{F}}{E} \qquad [13 \cdot 26]$$

If ρ is in circular mil ohms per foot, l_m in inches, $\mathfrak{F}$ in ampere-turns, and E in volts,

$$A = \frac{\rho l_m \mathfrak{F}}{12E} \text{ cir mils} \qquad [13 \cdot 27]$$

For copper wire at 60° C, $\rho_{60} = 12$ and we formulate

$$A = \frac{12 l_m \mathfrak{F}}{12E} = \frac{l_m \mathfrak{F}}{E} \text{ cir mils for } 60^\circ \text{ C copper} \qquad [13 \cdot 28]$$

Thus we observe that the size of wire for an electromagnet is not optional

but is definitely established by the ***resistivity*** and ***mean length per turn*** of the coil, by the ***mmf*** and by the applied ***emf***.

Example. An electromagnet requiring 12,000 ampere-turns is designed to have a mean length of turn of 10 in. and is to operate on 12 volts. Compute the size of copper wire required, assuming temperature 60° C.

By substituting in the derived relation (equation 13·28),

$$A = \frac{l_m \mathfrak{F}}{E} = \frac{10 \times 12{,}000}{12} = 10{,}000 \text{ cir mils}$$

This corresponds to a No. 10 wire.

13·9. Number of Turns. There is no simple relation to be formulated which will give the precise number of turns in the direct manner by which the wire size was determined.

Since at best something of a guess-and-test process is required, the factors which determine the number of turns may be most quickly disclosed by considering the consequence of using but *one turn* in the example just computed for wire size.

One turn of No. 10 copper wire of length $l_m = 10$ in. has resistance $R = \rho \dfrac{l}{A} = 12 \dfrac{10/12}{10{,}000} = 0.001$ ohm. The current produced in it by 12 volts is $I = E/R = 12/0.001 = 12{,}000$ amp. It is clear from previous consideration of the current-carrying capacity of wires that No. 10 wire would promptly vanish if subjected to this current. A rule of thumb commonly employed for small coils such as the one of our example is that from 1000 to 1500 circular mils per ampere will be found to produce allowable temperatures. This means that our current here must not exceed 10 amp and that the turns must not be less than $N = \mathfrak{F}/I = 12{,}000/10 = 1200$ turns.

For a more scientific analysis we must compute the heat-dissipating ability of the magnet (usually the coil surface) and use sufficient coil turns to keep the power taken by the magnet ($P = EI$ or $P = I^2R$) down to the value which can be dissipated as heat at the permissible temperature.

By substituting $I = \mathfrak{F}/N$ in $P = EI$ we find that the power input is expressed by

$$P = \frac{E\mathfrak{F}}{N} \qquad [13\cdot29]$$

Within the limited temperature range permissible for ordinary electrical wire insulation it is commonly acceptable to use the approximate heat transfer formula.

$$P = KSt \qquad [13\cdot30]$$

where S = heat dissipating surface (usually of the coil).
K = coefficient of heat dissipation for surface S.
t = temperature of surface S above the ambient.

Equating the input and output powers we find

$$P = KSt = \frac{E\mathfrak{F}}{N} \qquad [13\cdot31]$$

or

$$N = \frac{E\mathfrak{F}}{KSt} \qquad [13\cdot32]$$

The unit of area most used in design work is the square inch or square foot. Values of K commonly run not far from 0.01 watt per square inch-° C. Temperature rise t is usually specified as 40° to 50° C with hot-spot temperature 60° C.

It is necessary to *check the available space* provided for the coil to insure that the requisite number of turns can be accommodated. Computation of the space occupied by N turns should be an evident process, if care is exercised to provide for the requisite insulation as well as for the conductor metal. Space factors giving (area of copper)/(area occupied) are commonly provided in electrical handbooks.

Let it be clear that the above considerations merely establish the *minimum* number of turns required to insure continued operation of the electromagnet without *overheating*. For reasons reminiscent of those encountered in the selection of wire for electrical distribution there are factors other than temperature to be considered in a well-balanced design. In contrast with the distribution case, however, we must here adjust the amount of copper by altering the *length* instead of the *cross section* of the wire.

13·10. Design Considerations. The foregoing relations provide material for several worth-while observations regarding the design problems encountered in electromagnets and therefore in much electrical apparatus.

For the most economical design of magnet a guess-and-test method is usually required in order to obtain a coordination of coil, coil space, and coil temperature so that the coil will just fit the space and will take and dissipate just that power which will make the temperature close to the permissible limit. This is for economy of *material* so that the manufacturing cost may be a minimum consistent with reliable operation. There is no assurance that this will give the purchaser and operator the greatest *overall* economy of magnet—this because his *power bill* for

such a design *may* be unreasonable in comparison with the *purchase price* of the magnet. From this point of view Kelvin's law, as discussed for size of wire in power distribution, is equally applicable here. Especially for the larger magnets, this is important.

It may be observed that *higher voltage* in general requires both *smaller wire* and *more turns*, as well as *better insulation*. All these mean higher cost for materials and manufacturing operations. So long as the space factor remains unchanged, the same magnet structure will accommodate windings of different voltage rating without change in power, temperature, or ampere-turns provided the size of wire is correct and the same weight of copper is used.

It is well to observe that a larger and more powerful magnet, for a given shape, will *not* have a proportionally larger *surface*, i.e., surface does not increase in proportion to increase in *volume*. In general this means that a given design cannot simply be proportionately enlarged—it would run hot due to insufficient heat-dissipating surface. This behavior is not confined to magnets but applies to the design of electrical equipment in general.

13·11. Efficiency. It is to be noted that *lifting magnets*, such as those commonly used for handling sheet and pig iron, employ the magnetically produced force *not to displace* the load but merely to hold onto it in lieu of a hook, the *lifting* or displacement being done by a *crane*. Consequently the magnet produces only mechanical *force* and *no mechanical power;* the power input is converted *entirely* into *heat*. It follows that there is no such thing as a power or energy efficiency for the magnet. Nevertheless the term *efficiency* is used for magnets and can only mean *pounds pull per watt* which is not expressible in per cent.

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QUESTIONS

13·1. For a magnetic field of uniform density show that the energy per unit volume of the field is a simple function of flux density B and permeability μ.

13·2. What is the principle of virtual displacement and how is it used here?

13·3. What becomes of the electrical energy put into an electromagnet when it is *sustaining* a load (weight)?

13·4. Practical observation shows that the pull of a magnet depends greatly on the air gap d or the effective distance from the load. Explain why d does *not* appear in the pull relation

$$F = K\frac{\phi^2}{A}$$

There are *two* reasons.

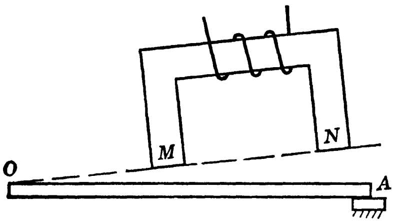
13·5. Your practical radio friend has a 110-volt (field coil) dynamic speaker which he wishes to use *permanently* on 220 volts. He states there is room to wind onto the present coil as much more wire of the *same size* as it now has. He inquires whether this will be all right, and if not, what should be done. Advise him fully.

13·6. What is *space factor* for a magnet winding?

13·7. Explain why fringing of flux is likely to have less effect on the pull of electromagnets than change of flux when the air gap is increased without changing the ampere-turns.

PROBLEMS

13·1. The structure here sketched is so designed that the armature OA pivoted at O, when raised by pulls at M and N will reduce the air gaps at both M and N to nearly zero. In the position shown the air gap at N is greater than that at M. Explain how the initial pulls at M and N are related if:



(*a*) No leakage flux and no fringing of air gap flux is assumed.

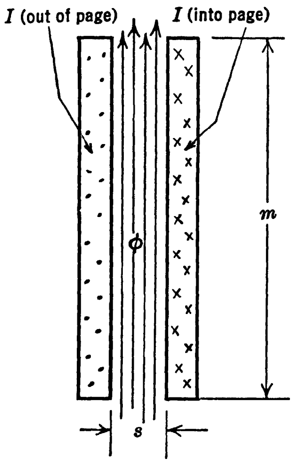
(*b*) Fringing of air gap flux is considered.

13·2. An electromagnet coil is required to produce 26,200 ampere-turns. It is wound with aluminum wire which has a resistance of 1.91 ohms per 1000 ft at the operating temperature of the coil. The mean length of turn is 24 in. The power dissipated in the winding is not to exceed 1 kw. Determine:

(*a*) Voltage to be applied to the coil.

(*b*) Number of turns.

(*c*) Will the power dissipated be greater (or less) if more turns are used with the same voltage as in (*a*)? Explain.



13·3. A small electromagnet coil having a mean turn length of 8 in. is to produce an mmf of 2000 ampere-turns. Assuming an operating temperature of 60° C and a current density of 1 amp per 1000 circular mils, determine the size of copper wire, the number of turns, and the exciting current for an applied voltage of (*a*) 10 volts, (*b*) 100 volts.

If the coil has an external radiating surface of 30 sq in., and a radiation constant of 0.01 watt per square inch for each degree centigrade rise is assumed, what will be the approximate temperature rise of the winding in each of the above cases?

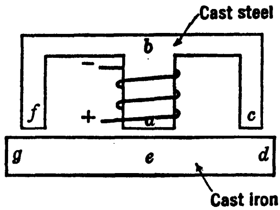
13·4. An end view of two bus bars carrying the same current I in opposite directions is as sketched

with spacing s and height (bar width) m. The magnetic field between bars may be assumed of uniform density and to be so much higher in density between bars than it is for the external portion of its path that all the available magnetic potential is utilized entirely for the internal portion of the path.

Derive an expression for the mechanical force of repulsion between bars *per meter of bar length* and compute its value in pounds (per meter) for $m = 16$ cm, $s = 3$ cm, and $I = 4000$ amp.

13·5. Given a lifting magnet as follows:

Segment	*Lengths*	*Areas*
ab	20 cm	16 sq cm
bc	40 "	10 " "
bf	40 "	10 " "
cd	0.1 "	10 " "
fg	0.1 "	10 " "
de	20 "	15 " "
ge	20 "	15 " "
ea	?	16 " "

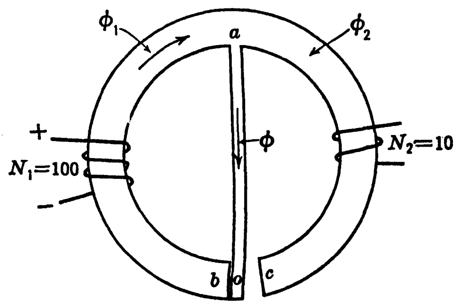


Flux $\phi_{ab} = 240$ kilolines.
Coil mmf $\mathfrak{F}_g = 5620$ ampere-turns.
Applied voltage $E = 110$ volts.
Mean length of turn $= 11$ in.
Permissible power loss in coil $= 220$ watts.
Operating temperature of coil is 60° C.

Determine:

(*a*) Path and direction of flux lines.
(*b*) Length of air gap *ae*.
(Tabulate all values involved in solution and write all equations clearly.)
(*c*) Per cent of the total pull exerted by an outer leg (*f* or *c*).
(*d*) Theoretical cross section of copper wire used for the coil.
(*e*) Number of turns on the coil.

13·6. The accompanying magnetic circuit is made entirely of cast steel. The ring section is 5 sq cm and that of the tongue *ao* is 1.0 sq cm. Length *ao* (approximately the mean diameter of the ring) is 8 cm. Assume lengths $ab = ac = 4\pi$ cm.



When the flux density in the tongue is 17 kilogauss the net magnetic force in the two gaps *bo* and *co* is twice that required to hold the tongue in contact with *b* (gap *bo* = zero) against the mechanical force of flexure of the tongue from mid-gap position. Coil N_2 has 10 turns carrying 20 amp and the flux ϕ_1 is 35 kilolines.

Neglect all leakage fluxes and fringing *except* in gap *oc* where the effective section may be assumed 10 per cent larger than that of the ring.

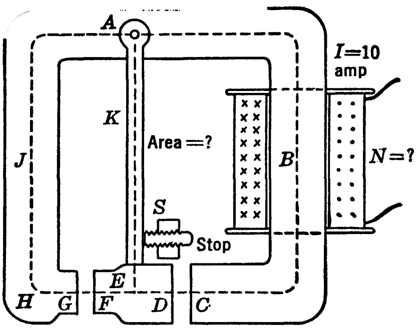
Under these conditions:

(*a*) How many amperes must N_1 carry?

(*b*) What air gap length *oc* must be provided?

(*c*) What is the mechanical force of flexure?

13·7. The accompanying figure shows part of the mechanism of a *magnetic relay.* For *small* values of exciting current, most of the flux crossing the air gap *CD* will pass through the lever *EKA* and back through the coil. This will cause *FD* to move to the *right* against the adjustable stop at *S*. For *large* values of exciting current the shunt *EKA* will become saturated, and then most of the flux crossing *CD* will pass on through the gap *FG*.



Part	Mat'l	Length cm	Sectional Area sq cm
ABC	CS	25	3
CD	Air	0.05	3
DE	CS	1	3
EF	CS	1	1.5
FG	Air	0.05	1.5
GH	CS	1	1.5
HJA	CS	15	3
EKA	CS	10	?

Since the magnetic pull depends not only upon the square of the flux but also inversely on the sectional area, the pull at *FG* may become *greater* than that at *CD*, even though ϕ_{FG} be less than ϕ_{CD}. This will cause *FD* to move to the left and close the gap *FG*. For some critical value of current, the pulls in the two air gaps are equal. This is an unstable equilibrium but marks the transition in the net force on the movable member from one direction to the opposite.

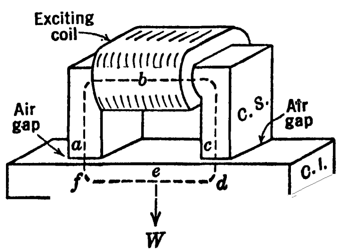
Determine:

(*a*) The cross section of the lever *EKA* such that the critical or operating current will occur when the flux density in the *main core ABC* is 12 kilogauss.

(*b*) The number of turns that must be placed on the exciting winding for a critical current of 10 amp.

13·8. The lifting magnet shown in the accompanying figure has a flux density of 15 kilogauss in the air gaps in order to sustain the weight *W*.

Path	Material	Length cm	Area sq cm
abc	Cast steel	40	250
def	Cast iron	30	1000
fa = *cd*	Air gap	0.1	250



Determine:

(*a*) The total pull in pounds exerted by the lifting magnet.

(*b*) The ampere-turns which must be supplied by the exciting winding.

(*c*) The commercial size wire to be used for the magnet coil if the mean length of turn is 71 cm and the voltage applied is 110 volts (60° C.).

(*d*) The number of turns that must be wound on the coil if the allowable power loss is not to exceed 55 watts.

(*e*) Solve parts (*c*) and (*d*) for an applied voltage of 220 and compare with the 110-volt coil.

13·9.

DESIGN OF LIFTING MAGNET

1. Total pull F for the two gaps.............................. 10,000 lb
2. Flux density B in air gaps.............................. ______ kl/sq in.
3. Ratio of A_1/A_2.............................. 1
4. Ratio of w/d.............................. ______
5. Ratio of h'/h.............................. 0.8
6. Thickness of steel plate (sheet steel) armature.............. 1.5 in.
7. Applied potential.............................. 220 volts
8. Space factor of winding.............................. 0.5
9. Length of each air gap.............................. 40 mils
10. Temperature rise to be between 40° and 50° C above ambient 20° C.
11. Heat dissipating coefficient of coil k = 6.5 milliwatts per sq in. per deg C.
12. Resistivity of copper = 12 cir mil ohms per ft at 60° C.
13. Weight of cast steel = 0.284 lb per cu in.
14. Weight of copper = 3.03×10^{-6} lb per cir mil ft.
15. Cost of cast steel = $0.05 per lb.
16. Cost of copper magnet wire = $0.25 per lb, approximately (varies with size and market).

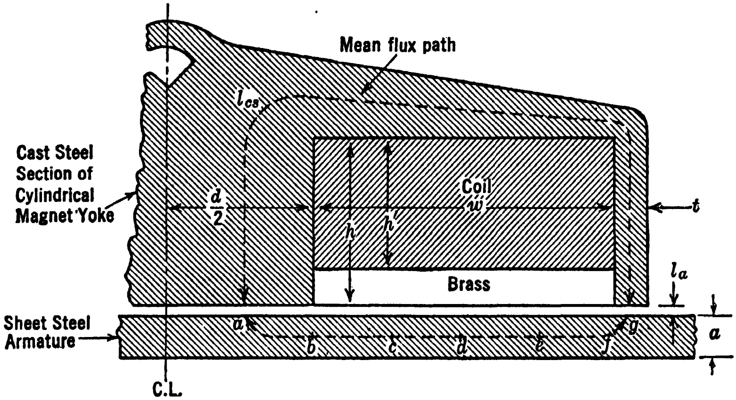
Values to be assigned by instructor:

Data	*B*	*w/d*	*Data*	*B*	*w/d*	*Data*	*B*	*w/d*
1	70	0.3	10	85	0.6	16	95	0.8
2	70	0.4	11	85	0.7	17	95	0.9
3	70	0.5	12	85	0.8	18	95	1.0
4	75	0.4	13	90	0.7	19	100	0.9
5	75	0.5	14	90	0.8	20	100	1.0
6	75	0.6	15	90	0.9	21	100	1.1
7	80	0.5						
8	80	0.6						
9	80	0.7						

Assume

17. *Negligible fringing and leakage.*
18. *Uniform flux density throughout cast steel yoke.*
19. *Ratio of* $w/h = 2$ (to be changed later if necessary).

Plan view consists of concentric circles centered on the dot and dash center line, and having radii conforming to the horizontal dimensions given in section below.



Method of Solution

From the initial assumptions determine:

(*a*) ΣHl for air gap and yoke.

(*b*) ΣHl for armature. Plot $H = f(l)$ and find area under curve as in Problem 3 of Chapter XII.

(*c*) Necessary commercial size of wire (A.W.G.)

(*d*) Temperature rise in degrees centigrade.

If the temperature rise does not fall within the prescribed limits redesign the magnet by varying dimension h until it does.

In so far as the effect of a change in h on the required ampere turns may be neglected it should be observed that change in h produces a directly proportional change in *number of turns* and in *coil resistance*. Power $P = E^2/R$ therefore is *inversely proportional* to h. An expression for heat dissipating surface in terms of h is readily set up. The effect of h on the power per unit of heat-dissipating area, and therefore on the coil temperature, is readily obtained to guide the selection of a new value of h.

(*e*) For final design find:

(1) Cost of cast steel yoke.

(2) Cost of copper.

(3) Total cost of materials used.

Design of Lifting Magnet

For air gap, B = ________________ kl per sq in. $\frac{w}{d}$ = ________________

	Trial	Final		Trial	Final
$F_1 = F_2$ (lb)			*Mmf* (*Hl's*) (*continued*)		
$A_1 = A_2$ (sq in.)			Armature (from graph) (amp-turns)		
d (in.)			Total (amp-turns)		
w (in.)			l_m (length of mean turn, in.)		
$D = d + 2w$ (in.)			ρ (copper at 65° C) (cir mil ohms per ft)		
$D_m = d + w$ (in.)			E (volts)		
t (in.)			A (computed), (cir mils)		
h (in.)			A.W.G. wire		
$h' = 0.8h$ (in.)			A (actual), (cir mils)		
a (in.)			Coil section ($w \times h'$), (in^2)		
B_{cs} (yoke) (kl per sq in.)			Space factor		
ϕ (kl)			Copper section, (cir mils)		
Armature densities (plot)			Coil turns		
B_a (kl per sq in.)			R for coil (actual) (ohms)		
B_b (kl per sq in.)			I (amp)		
B_c (kl per sq in.)			P (watts)		
B_d (kl per sq in.)			*Cooling surface*		
B_e (kl per sq in.)			Cylindrical coil surface (sq in.)		
B_f (kl per sq in.)			1 Coil end (sq in.)		
B_g (kl per sq in.)			Total (sq in.)		
Armature gradients (plot)			Watts per sq in. (to be dissipated)		
H_a (amp-turns per in.)			Cooling coefficient watts per in^2-°C		
H_b (amp-turns per in.)			Temp. rise of coil surface °C		
H_c (amp-turns per in.)			Volume of CS, cu in.		
H_d (amp-turns per in.)			Weight of CS, lb		
H_e (amp-turns per in.)			Cost per lb of CS	Compute only for final design	
H_f (amp-turns per in.)			Cost of CS		
H_g (amp-turns per in.)			Length of wire (ft)		
Average gradients			Weight of copper		
Yoke (amp-turns per in.)			Cost per lb of copper		
Air gaps (amp-turns per in.)			Cost of copper		
Mean Flux Path Lengths			Total cost of metal		
Yoke (in.)					
Air gaps (total) (in.)					
Armature (in.)					
Mmf (*Hl's*)					
Yoke (amp-turns)					
Air gaps (total both) (amp-turns)					

Computed by.

Chapter XIV

INDUCTANCE

14·1. Self-Induction of Emf. In electromagnetic induction of emf $e = N(d\phi/dt)$, one basic case, that of *transformer action,* involves no mechanical motion; the flux is changed by changing the mmf. The factors involved are sketched in Fig. 14·1 for the condition where an increasing current through coil A produces an increasing flux through coil B and induces emf e with the polarity shown.

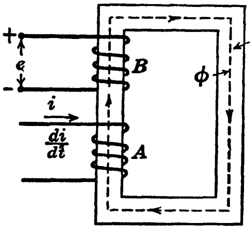
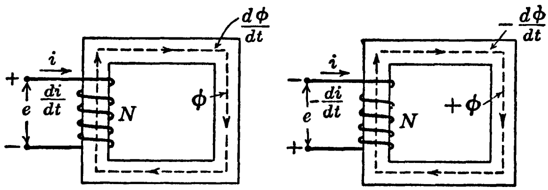


FIG. 14·1.



(a) (b)

FIG. 14·2.

Let us now observe clearly that the changing flux links with coil A as well as B. If our analysis of electromagnetic induction was correct, an emf should be induced in coil A as well as in B. This is indeed the case, as shown in Fig. 14·2*a* for an *in*creasing current and in 14·2*b* for *de*creasing current in coil A. For this special case of induction, appropriately called *self-induction,* it is possible and useful to express the emf in terms of the *current* instead of the flux, as follows.

From magnetics,

$$\phi = \mathcal{P}\mathfrak{F} = \mathcal{P}Ni = N\mathcal{P}i,$$

Differentiating,

$$\frac{d\phi}{dt} = N\left(\mathcal{P}\frac{di}{dt} + i\frac{d\mathcal{P}}{dt}\right)$$

Induced emf,

$$e = N\frac{d\phi}{dt} = N^2\left(\mathcal{P}\frac{di}{dt} + i\frac{d\mathcal{P}}{dt}\right) \qquad [14\cdot1]$$

When $\mathcal{P}\left(= \mu\dfrac{A}{l}\right)$ is *constant*

$$e = N^2\mathcal{P}\frac{di}{dt} \qquad [14\cdot2]$$

Now let us replace the constants N and $\mathcal{P}$ by a single factor

$$L = N^2\mathcal{P} \qquad [14\cdot3]$$

which gives

$$e = L\frac{di}{dt} \qquad [14\cdot4]$$

This quantity L is called ***inductance*** and is defined simply as ***the ratio of the self-induced emf to the time rate of change of the inducing current.***

$$L = \frac{e}{di/dt} \qquad [14\cdot5]$$

14·2. Dimension and Unit of Measure. From $L = N^2\mathcal{P}$ it is evident that the dimension of L and $\mathcal{P}$ are the same. In the *practical system* of dimensions this is $[RT]$ and is easily checked through the relation $L = \frac{e}{di/dt} = [EI^{-1}T] = [RT]$.

It does *not* follow that the *unit of measure* of L and $\mathcal{P}$ are the same, however. The N^2 factor, while dimensionless, is a significant quantity measured in $(\text{turns})^2$ and cannot properly be ignored here any more than turns can be ignored in the ampere-turn unit for the measure of mmf. Particular mention is made of this because it is not uncommon in the literature to find that the same unit of measure is used for both L and $\mathcal{P}$. As earlier mentioned, permeance $\mathcal{P}$ is without its own unit of measure and must be expressed in MKS webers per ampere-turn or the equivalent units of other systems of measure. Only MKS units will be used in this study.

Inductance is measured in ***henrys***. An inductance of one henry will have one volt of emf for a current-change of one ampere per second. It follows that permeance might be expressed in henrys per turn^2. This is not being done, however, in practice.

14·3. Other Inductance Relations. We have seen in the study of resistance that R may be determined from either of at least two relations; one a *functional* or operating relation $R = E/I$; the other a *constructional* relation $R = \rho(l/A)$ which reveals the influence of dimensions and composition on resistance as a physical entity. In like manner it is to be observed that $L = \frac{e}{di/dt}$ and $L = N^2\left(\mathcal{P} = \mu\frac{A}{l}\right)$ respectively portray for inductance its function or business, and its construction or composition and shape. The one relation enables the quantity to be measured in *action* or its action predicted, while the other relation enables computation of the quantity as a possibly dormant *structural entity.*

Thus we conceive for inductance the same sense of physical reality and substance that we conceive for resistance; the mathematician terms such quantities parameters and it is not unusual to find R and L called electric circuit parameters.

Other rather basic relations for inductance are useful. Among these should be mentioned one which is readily developed as follows.

$$L = N^2 \mathcal{P} = N^2 \frac{\phi}{\mathfrak{F}} = N^2 \frac{\phi}{NI} = \frac{N\phi}{I} = \frac{\lambda}{I}$$

$$= \frac{\text{Flux linkages (weber-turns)}}{\text{Current (amperes)}} \qquad [14\cdot6]$$

In consequence of this relation inductance is sometimes referred to as being *flux linkages per unit current* and *weber-turns per ampere.* While the constructional reality of inductance may seem less clear in this concept it is very useful for the analysis and computation of some types of problems, especially where field mapping is used to find the flux linkages.

14·4. Energy Stored by Inductance. It has been observed that a magnetic field or flux represents stored energy. The magnetic flux which links an inductance is no exception and the energy frequently is said to be *stored in the inductance* when the electric rather than the magnetic circuit is receiving major attention. The energy is expressible in terms of the inductance and current exclusively, as may be shown by any of the following derivations.

$$W = \int e\,dq = \int ei\,dt = \int \left(L\frac{di}{dt}\right) i\,dt = L\int i\,di = \tfrac{1}{2}Li^2 \qquad [14\cdot7]$$

$$W = \int p\,dt = \int ei\,dt = \int \left(L\frac{di}{dt}\right) i\,dt = L\int i\,di = \tfrac{1}{2}Li^2 \qquad [14\cdot8]$$

$$W = \int \mathfrak{F}\,d\phi = \int \mathfrak{F}\mathcal{P}\,d\mathfrak{F} = N^2\mathcal{P}\int i\,di = L\int i\,di = \tfrac{1}{2}Li^2 \qquad [14\cdot9]$$

14·5. Polarity-Direction Relations. The relation $e = L(di/dt)$ has been developed without particular concern about the algebraic sense of its several factors. Because the basic $e = N(d\phi/dt)$ is commonly developed with an algebraic sense, $e = -N(d\phi/dt)$, it is not uncommon to carry the minus sign over into self-induction and write $e = -L(di/dt)$. This is not at all necessary and is more often misleading than otherwise. Examination of the phenomenon physically rather than mathematically will make the issue clear.

In the basic induction phenomena symbolized by $e = N(d\phi/dt)$ the relation among polarity of emf, direction of flux, and flux change is

complicated because the geometry of the flux linking is in a right angle sense and calls for special definition of direction as developed in Chapter V with the aid of a right-hand rule.

In self-induction it is unnecessary to pursue the reasoning required for the general case of induction. Here the emf and current of $e = L(di/dt)$ are directly related within the electric circuit in manner similar to the emf and current of Ohm's law $e = Ri$. Consideration of the flux in any way or of a right-hand rule is then wholly unnecessary for determining the relation between polarity of emf and direction of current.

The best approach to the relation is through the energy aspect of the phenomenon.* We have seen that the stored energy is $W = \frac{1}{2}Li^2$. From this it is clear that *in*creasing current entails *in*creasing stored energy and that inductance, to store energy, must function as a load or energy-receiving element during increase of current.

Now we know that through a *load* element of any kind, current must flow from positive to negative as through a resistance. It follows that:

1. ***Increasing current flows through inductance from positive to negative.*** (See Fig. 14·2*a*.)

Similar reasoning discloses that, for decreasing current, inductance functions as an energy source and that:

2. ***Decreasing current flows through inductance from negative to positive.*** (See Fig. 14·2*b*.)

It should now be clear that, consistent with the writing of Ohm's law $e = Ri$, which always denotes *fall* of potential in the direction of current flow, we properly write $e = L(di/dt)$ (without minus sign) which denotes *fall* of potential in the direction of current flow when *current is increasing* $(+di/dt)$. Because of the additional factor, current change, it is to be observed that the *inductance law* has possibilities beyond the *resistance (Ohm's) law* as regards variation in the polarity of the emf.

* *Warning:* The reader will find in the literature that the polarity-direction relation for inductance is commonly developed by means of a corollary of the law of conservation of energy known as Lenz's law. This merely observes that the polarity of a self-induced emf is such as to oppose the change in the associated current. Clearly it is still up to the student to figure out the actual polarity of emf. The wrong polarity has resulted so often from this attack that it is not to be recommended in comparison with the direct consideration of energy employed in the above analysis.

To add to the unnecessary confusion it is also customary when writing $e = -L(d\phi/dt)$ (with the minus sign) to "explain" the minus sign by stating Lenz's law with the implication that *oppose* means *minus*. All voltages or emf's require "opposition" in order to exist; to assign minus signs only by this criterion would indeed be unwise. Note, for example, that we should then write $e = -Ri$ because this voltage certainly "opposes the current" as much as any induced emf ever will!

It is helpful possibly to observe that displacement of electricity through an inductance behaves like cars on a roller coaster, accelerating down hill (through fall of potential) and decelerating up hill (through rise of potential). In any case the student should strive to learn the facts through understanding of the phenomenon rather than by rote memory of statements 1 and 2.

14·6. Electrical Mass. Because certain electrical and mechanical relations are mathematically identical, a helpful analog between electrical inductance and mechanical mass is indicated as follows.

$$E = L\frac{di}{dt} \qquad\qquad F = M\frac{dv}{dt}$$

$$W = \tfrac{1}{2}Li^2 \qquad\qquad W = \tfrac{1}{2}Mv^2$$

Perhaps the most helpful feature of the analog is to clarify the relation between e and i of $e = L(di/dt)$ as regards *cause* and *effect*. Consider first the analogous $F = M(dv/dt)$. We are inclined to say that *force F causes* mass M to be accelerated in the amount dv/dt—this to describe possibly the pull of a locomotive on a train when getting underway. On the other hand, if the locomotive collides with an obstacle, we explain the changed dimensions of either participant by saying that the *deceleration dv/dt causes* a force F to be exerted on them. In other words cause and effect in this case associate with force F and change of velocity dv/dt according to point of view rather than by any fundamental relation. We might, after the fashion of the chemist, write $F \rightleftarrows M(dv/dt)$ (read "F produces and is produced by dv/dt of M"), or we might be inclined to chatter about forces of action and reaction with appropriate definitions and restrictions. Whatever mathematical or grammatical mechanism we invoke for the phenomenon, it stands that *force* and *change of motion* are *mutually causative*, the one of the other—simply coexistent.

Exactly the same situation exists for e and i of $e = L(di/dt)$. While admittedly we have found it convenient to develop the relation in terms of induction of e as though caused *by di/dt*, it is equally tenable to conceive that the emf *causes* the change in current. Quite as it requires time to accelerate a body to some steady velocity, so it requires time to bring current flow to some steady value.* When emf is applied to an electric circuit as by switching, the current does not reach its value

* It is not to be presumed that this constitutes acceleration in terms of *electron* mass as discussed in Chapter III for the free electrons of a cathode ray. The kinetics of metallic conduction is more complicated, involving but very low velocities of current, and must here be considered in terms of mass only as represented by *inductance*.

instantaneously, and it is entirely proper to say that the emf not only causes the ultimate current but also *causes the preliminary growth* of current.

It is wholly unnecessary to invoke notions of *applied* and *induced* or *back* emf so long as *polarity* rather than direction of emf is recognized. An emf between two points *can have but one polarity* at any one time—one point is + and the other − but the directions which may be imagined have, like forces, a working minimum of two (action and reaction) and no limiting maximum. It is important to appreciate that emf, in contrast with force, is so constituted as to offer this really significant and labor-saving advantage of unambiguous simplicity. Because our pioneering ancestors became altogether excusably entangled in an artificial argument about "emf into" versus "emf out of," "applied emf" versus "back emf," or "counter emf," or "induced emf," and many more, is no excuse for it being continued into the light of today. Let us bear in mind that between two points there can be at any time but *one* difference of potential with but one polarity, the one point + and the other −, with no more direction than the hot and cold of a temperature difference in Ohm's thermal analog.

14·7. Danger, High Voltage! Because a reasonable force is used to accelerate a body is certainly no assurance that an unreasonably large force may not be produced by a deceleration of the body as in collision. The body possesses kinetic energy $W = \frac{1}{2}Mv^2$ which must be disposed of by the body, however short the time for it may be. In terms of distances through which the energy is expended, the force is $F = dW/ds$ or the average force $F_{avg} = W/s$. During a collision the force, or rate of energy dissipated per foot of motion, may be enormous.

In the same sense the emf applied to an inductance may be very moderate, say 6 volts, and require an observable time for growth of the current to ultimate steady value. If the electron flow (or current) is abruptly stopped, as by simply breaking the circuit, the deceleration is so large that the emf $e = L(di/dt)$ may easily run into thousands of volts. In terms of energy, the kinetic energy $W = \frac{1}{2}Li^2$ must be disposed of by the time the current becomes zero. Breaking the circuit in effect produces a "collision," as though the electrons collided with the broken end of the wire so violently as to produce an emf high enough to break down the air gap and form an arc in which the requisite energy is dissipated as heat. Precisely this process was long used for an ignition system on many gasoline engines, especially marine motors. It was called the *make-and-break* or low-tension system and was less vulnerable to failure from moisture before better insulating materials were developed for the high-tension or jump-spark ignition now prevalent.

It is necessary to be on the alert for emf's of self-induction even with seemingly quite harmless electrical apparatus. Insulation may be broken down and measuring instruments ruined even if life is not jeopardized. In electrical machine laboratories there is a temptation impulsively to open the always highly inductive shunt field circuit of a motor or generator before disconnecting a voltmeter from across the field terminals—the meter depreciation, while impressive, is preferably avoided. Some field circuits are arranged to be opened in sections to protect the insulation; others are arranged so that resistance is automatically inserted in the field circuit before the actual break can occur. The resistance "puts on the brakes" for the current before "hitting the wall."

Let it be clear that to *anticipate the consequence* is in general quite as important for the *opening* as for the *closing* of an electric circuit.

14·8. Inductance with Variable Permeance. In all the foregoing it has been emphasized from the beginning that permeance must be constant and that inductance must *not* be a function of current or flux. It would be wishful idealism however to ignore the practical need for extending the concept of inductance beyond the realm of constant permeance. This arises from the prevalence of ferromagnetic circuits with their typically nonlinear characteristics. Let us consider the consequences.

We shall stand fast on the definition of inductance represented by equation 14·5.

$$L = \frac{e}{di/dt} \qquad [14\cdot5]$$

By equating this voltage with that of the basic induction relation (equation 14·1) from which it was derived we obtain

$$e = L\frac{di}{dt} = N\frac{d\phi}{dt}$$

or

$$e\,dt = L\,di = N\,d\phi = d\lambda \qquad [14\cdot10]$$

and

$$L = N\frac{d\phi}{di} = \frac{d(N\phi)}{di} = \frac{d\lambda}{di} \qquad [14\cdot11]$$

Comparing this with equation 14·6 ($L = \lambda/I$) we conclude that inductance is represented in general by ***change in flux linkages per unit change in current*** rather than by simply flux linkages per unit current, λ/I. To

understand the practical distinction, reference to a plot of λ versus i is helpful. From magnetic circuit relations we find

$$\phi = BA$$

or

$$\lambda = N\phi = (NA)B$$

and

$$Ni = \mathfrak{F} = Hl$$

or

$$i = \left(\frac{l}{N}\right)H$$

With NA and l constant it is clear that

(*a*) flux linkages $\lambda = N\phi$ is directly proportional to B.

(*b*) current i is directly proportional to H.

The λ versus i plot then is simply the familiar B-H curve with new scales as shown in Fig. 14·3. For constant μ materials the graph is

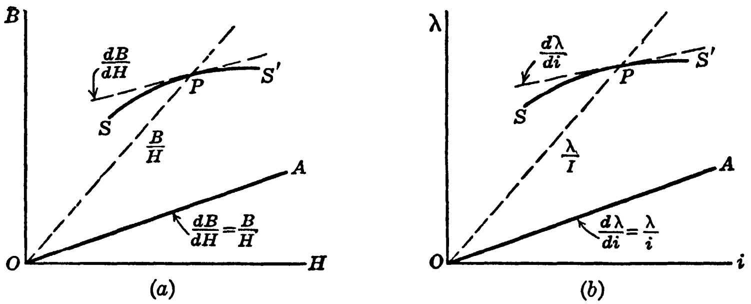


FIG. 14·3.

linear (OA) and the slope is the same as the ratio of the coordinates at any point.

$$\frac{dB}{dH} = \frac{B}{H} \quad \text{and} \quad \frac{d\lambda}{di} = \frac{\lambda}{I}$$

These relations however apply *only* to the *linear* graph. For ferromagnetic curve SS', $(\lambda/I) \neq (d\lambda/di)$; the ratio of the coordinates of point P is not the same as the slope of the curve at point P. Ratio λ/I is merely the slope of line OP joining P to origin O. Since only $d\lambda/di$ will satisfy the definitive equations 14·5 and 14·11 it becomes evident that for saturable materials λ/I is *not* applicable; inductance computed as λ/I will *not* give the true value of induced emf in $e = L(di/dt)$. In the same way $B/H \neq dB/dH$, and permeability $\mu = (B/H)$ is *not* constant

and is *not* applicable to the induction phenomena for computing $\mathcal{P} = \mu(A/l)$ in $L = N^2\mathcal{P}$ or for L in $e = L(di/dt)$.

To meet this situation a different kind of μ is conceived and defined to be exempt from the constancy requirement. Because this new quantity is represented by the *differential* dB/dH rather than the *ratio* B/H, it is known as ***differential permeability*** and is usually subscripted d to distinguish it from plain invariable μ, thus:

$$\mu_d = \frac{dB}{dH} \tag{14·12}$$

Remembering that for plain inductance $L = N^2\mathcal{P}$, we find it possible to extend the new concept to permeance and to inductance to provide

$$\mathcal{P}_d = \frac{d\phi}{d\mathcal{F}} = \mu_d \frac{A}{l} \tag{14·13}$$

$$L_d = N^2\mathcal{P}_d \tag{14·14}$$

Thus we provide for the variable inductance case by creating *differential* permeability, *differential* permeance, and finally *differential* inductance. Although care is not always used to apply this distinction to the latter, equation 14·11, strictly speaking, should be written with the subscript.

$$L_d = \frac{d\lambda}{di} \tag{14·15}$$

This is well understood in the profession but instances of confusion due to laxity in clearly identifying this quantity are not rare in the literature.

14·9. Energy for Differential Inductance. Unfortunately the differential concept above described fails to carry through so simply for the energy relation. Returning to equation 14·7 and equation 14·10 we readily obtain

$$W = \int ei\,dt = \int i(edt) = \int i\,d\lambda \tag{14·16}$$

This, of course, is represented by the area to the left of the λ-i curve as in Fig. 14·4.

For the linear case, OA of Fig. 14·4a, the triangular area gives the familiar result

$$W = \frac{1}{2}\lambda I = \frac{1}{2}\left(\frac{\lambda}{I}\right)I^2 = \frac{1}{2}LI^2$$

For the ferromagnetic case, if we consider a normal magnetization curve such as OS (Fig. 14·4b), it becomes evident that the area is no

longer $\frac{1}{2}LI^2$, that $\frac{1}{2}L_dI^2$ is meaningless, and that we may either accept the graphical area for representing and computing W or we may write

$$W = \int L_d i \, di \qquad [14\cdot17]$$

Since we cannot satisfactorily express $L_d = f(i)$ analytically, this latter relation is of little utility and we are left only with the graphical computation.

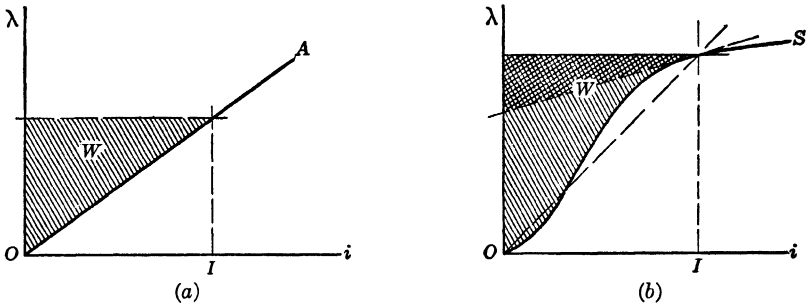


FIG. 14·4.

It should be noted further that the normal magnetization curve of Fig. 14·4*b* is by no means typical of what may be encountered for the λ–i curve or its corresponding *B-H* graph. For cyclic variation of i a hysteresis loop will appear and considerable care is required to designate the energy elements without confusion.

14·10. Incremental Inductance. Ferromagnetic inductances are frequently operated with a pulsating current of small amplitude relative

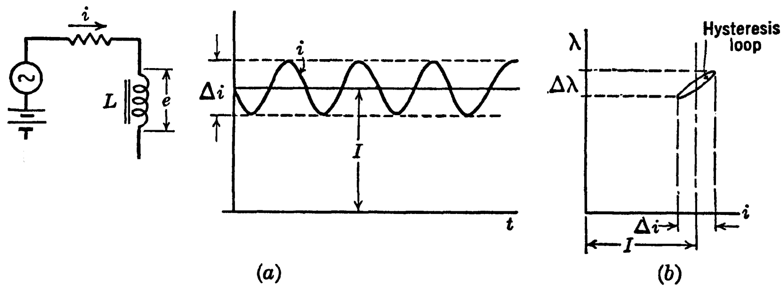


FIG. 14·5.

to the average value. These will be familiar to some as ***filter chokes.*** The λ–i (and *B-H*) graph will then appear like Fig. 14·5 and it becomes expedient to utilize an *approximation* of the exact differential concepts

in the form of finite increments to give the *incremental* quantities (note subscript i for incremental).

$$\mu_i = \frac{\Delta B}{\Delta H} \qquad [14\cdot18]$$

$$\mathcal{P}_i = \mu_i \frac{A}{l} = \frac{\Delta\phi}{\Delta\mathcal{F}} \qquad [14\cdot19]$$

and

$$L_i = N^2 \mathcal{P}_i = \frac{\Delta\lambda}{\Delta i} \qquad [14\cdot20]$$

Care is required to indicate either the *mean* value I or the *actual limiting* values of i rather than just Δi in cases of this kind. How far to permit the increments to extend is a matter of judgment for the specific case. Between plain L and differential L_d there will be found numerous practical possibilities beyond the bounds of treatment here.

14·11. Mutual Inductance. Let us now return to the two coils with which we opened the discussion of self-induction. It is useful to extend the concept of inductance, $L = \dfrac{e}{di/dt}$, which we have developed for the special case of self-induction, to include the case where emf e_2 is induced in one circuit element N_2 by current change di_1/dt in another circuit element N_1 as shown in Fig. 14·6. The ratio of emf e_2 to current change di_1/dt is termed mutual inductance, L_M.* The relation, of course, is

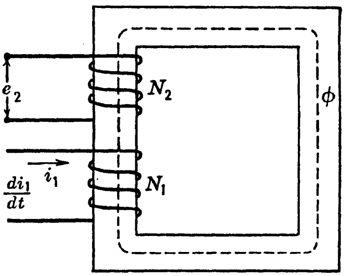


FIG. 14·6.

$$L_M = \frac{e_2}{di_1/dt} \qquad [14\cdot21]$$

The unit of measure and the dimension of mutual inductance are necessarily the same as for self-inductance. There are certain basic differences between the phenomena of self-induction and *mutual induction* which are noteworthy, however:

1. The relation $e_2 = L_M(di_1/dt)$, in contrast with that for self-induction, is a one-way reaction. The emf e_2 is caused *by* the di_1/dt

* The symbol L_M is used in this text rather than the more prevalent symbol M because official transfer of the latter to other use seems imminent. It is recognized that mutual inductance, being merely one member of the inductance family, needs no wholly separate symbol. When necessary the identity of *self*-inductance is readily emphasized by appropriate subscript such as L_s, L_{11}, or L_{22}, which are later used here.

and cannot cause di_1/dt because it cannot directly cause current to flow in N_1.

2. In contrast with L, which can have significance only as a plus algebraic quantity, mutual inductance L_M can have significance as a minus quantity. This will become apparent when an electrical connection is made between N_1 and N_2, either series or parallel, later in the discussion.

14·12. Other Relations for Mutual Inductance. Given the original relation

$$e_2 = N_2 \frac{d\phi}{dt} \qquad [14\cdot22]$$

Where the flux is

$$\phi = \mathcal{P}\mathfrak{F}_1 = \mathcal{P}N_1 i_1 \qquad [14\cdot23]$$

Differentiating gives

$$\frac{d\phi}{dt} = N_1\left(\mathcal{P}\frac{di_1}{dt} + i_1\frac{d\mathcal{P}}{dt}\right) \qquad [14\cdot24]$$

Assuming *constant* permeance, we have

$$\frac{d\phi}{dt} = (\mathcal{P}N_1)\frac{di_1}{dt} \qquad [14\cdot25]$$

Multiplying by N_2, the emf is

$$e_2 = N_2(\mathcal{P}N_1)\frac{di_1}{dt} \qquad [14\cdot26]$$

Then by definition

$$L_M = \frac{e_2}{di_1/dt} = N_1 N_2 \mathcal{P} \qquad [14\cdot27]$$

Like the $L = N^2\mathcal{P}$ for self-inductance, $L_M = N_1N_2\mathcal{P}$ shows that mutual inductance is wholly determined by the *composition* and *structure* of the magnetic and electric circuits involved.

This relation has a further important significance. Up to this point discussion has concerned current in N_1 and emf in N_2. It is clear now that $L_M = N_1N_2\mathcal{P}$ would have resulted also if the operation had been switched to make the current concern coil N_2 and the emf coil N_1. This, of course, depends upon the permeance $\mathcal{P}$ of the mutual flux path being the same in either case.

Mutual inductance is also expressible in terms of *flux linkages per unit current.* This may be shown by substituting $\mathcal{P} = \phi/\mathfrak{F}_1 = \phi/N_1i_1$ in equation 14·27.

$$L_M = N_1N_2\mathcal{P} = N_1N_2\frac{\phi}{N_1i_1} = \frac{N_2\phi}{i_1} \qquad [14\cdot28]$$

As might be expected, the relation involves linkage of flux with the one coil (N_2) and current in the *other* coil (N_1).

14·13. Inductance—Self, Mutual, and Nonmutual. It is rare that coils or circuit elements are so arranged that even approximately all the

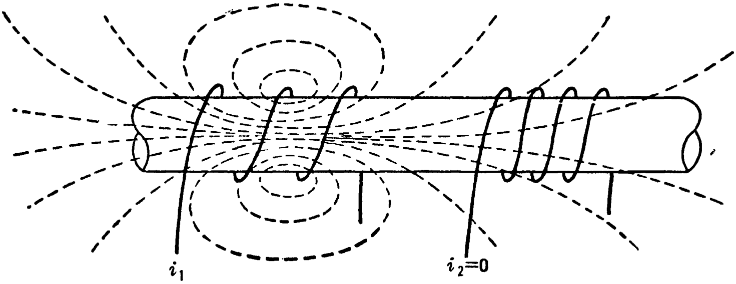


FIG. 14·7.

flux will be mutual flux as represented in Fig. 14·6. The actual condition is likely to be as suggested in Fig. 14·7 which assumes nonmagnetic materials.

Here we see that in addition to the strictly mutual and nonmutual fluxes there is a considerable assortment of flux lines linked with less than all the turns of one or both coils. These latter are known as ***partial linkages.*** They need not concern our present study because the effect of the actual fluxes can be accurately represented, as in Fig. 14·8*a*, by two *equivalent* fluxes ϕ_{12} and ϕ_{11} which we shall call, respectively, the ***mutual flux*** and the ***nonmutual flux*** for coil N_1. The flux subscripts ϕ_{12} and ϕ_{11} are

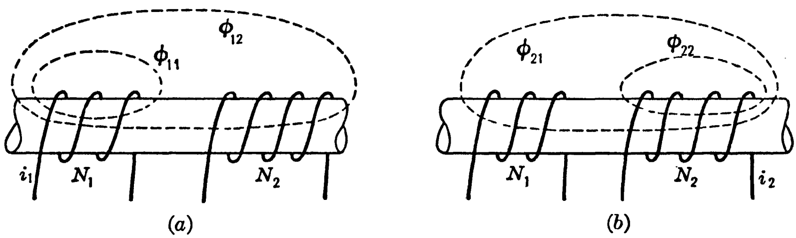


FIG. 14·8.

interpreted to read, respectively, "flux due to coil 1 which links coil 2" and "flux due to coil 1 which links only with coil 1."

These fluxes are equivalent to the actual fluxes on the basis that they will induce in each coil the same emf's as would the actual fluxes when the mmf of coil N_1 is changed by a current change di_1/dt.

The total flux linking coil N_1 is then expressed by

$$\phi_1 = \phi_{11} + \phi_{12} \qquad [14\cdot29]$$

Introducing now the mmf, $\mathfrak{F}_1 = N_1 i_1$, which produces this flux, the permeances of the respective flux paths are related as follows by substitution in equation 14·29:

$$\mathcal{P}_1 \mathfrak{F}_1 = \mathcal{P}_{11} \mathfrak{F}_1 + \mathcal{P}_{12} \mathfrak{F}_1$$

or

$$\mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{12} \qquad [14\cdot30]$$

From equation 14·3 the *inductance*, usually called the *self-inductance*, of coil N_1 is

$$L_1 = N_1^2 \mathcal{P}_1 \qquad [14\cdot31]$$

Likewise the inductance associated with the *exclusively* self-linking flux, ϕ_{11}, which we shall call the ***nonmutual inductance*** is

$$L_{11} = N_1^2 \mathcal{P}_{11} \qquad [14\cdot32]$$

From equation 14·27 the ***mutual inductance***, where $L_M = L_{12}$, is

$$L_{12} = N_1 N_2 \mathcal{P}_{12} \qquad [14\cdot33]$$

It is important to note that

$$L_1 \neq \text{(is \textit{not} equal) } L_{11} + L_{12}$$

To determine the correct relation we substitute equation 14·30 in equation 14·31 to give

$$L_1 = N_1^2 \mathcal{P}_{11} + N_1^2 \mathcal{P}_{12} = N_1^2 \mathcal{P}_{11} + \frac{N_1}{N_2}(N_1 N_2 \mathcal{P}_{12})$$

or, using equations 14·32 and 14·33 gives

$$L_1 = L_{11} + \frac{N_1}{N_2} L_{12} \qquad [14\cdot34]$$

If now coil N_2 carries current instead of coil N_1, as in Fig. 14·8*b*, we write in the same manner as previously,

$$\phi_2 = \phi_{22} + \phi_{21} \qquad [14\cdot35]$$

$$\mathcal{P}_2 \mathfrak{F}_2 = \mathcal{P}_{22} \mathfrak{F}_2 + \mathcal{P}_{21} \mathfrak{F}_2$$

$$\mathcal{P}_2 = \mathcal{P}_{22} + \mathcal{P}_{21} \qquad [14\cdot36]$$

$$L_2 = N_2^2 \mathcal{P}_2 \qquad [14\cdot37]$$

$$L_{22} = N_2^2 \mathcal{P}_{22} \qquad [14\cdot38]$$

$$L_{21} = N_1 N_2 \mathcal{P}_{21} \qquad [14\cdot39]$$

and

$$L_2 = L_{22} + \frac{N_2}{N_1} L_{21} \qquad [14\cdot40]$$

Now let currents i_1 and i_2 exist simultaneously. Because *constant permeability* is postulated, the fluxes ϕ_1 and ϕ_2 as well as their components may be superimposed, according to the principle of superposition, as shown in Fig. 14·9.

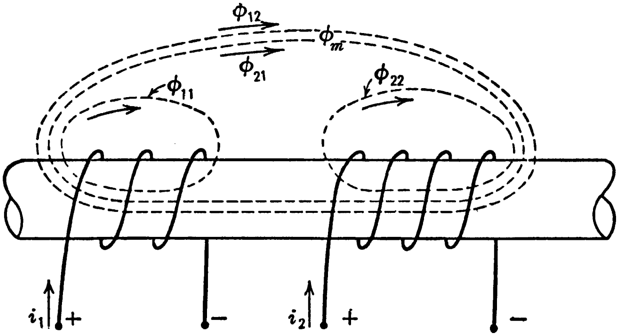


FIG. 14·9.

The nonmutual fluxes ϕ_{11} and ϕ_{22} remain as before and the mutual flux is now the summation of two components.

$$\phi_M = \phi_{12} + \phi_{21} \qquad [14·41]$$

14·14. Energy Shows $L_{12} = L_{21}$. The question remains as to the relation between L_{12} and L_{21}. This is best established by means of a study of the energies stored in the magnetic fields or the inductances of the two coils. We have already observed that, for a single isolated coil, the field energy is $W = \frac{1}{2}LI^2$. The presence of coil N_2, inactive as in Fig. 14·8*a*, can have no effect on the energy stored in the field or inductance of N_1 while it is carrying current I_1, and we write

$$W_1 = \tfrac{1}{2}L_1I_1^2 \qquad [14·42]$$

If now, while keeping I_1 constant, current i_2 is brought from zero to value I_2, as in Fig. 14·9, the same flux ϕ_2 as in Fig. 14·8*b* will be produced and superimposed on ϕ_1, independent of the presence of the constant ϕ_1. Superposition of fluxes is permissible here because we are considering only the case for *constant permeability*. The energy put into coil N_2 is, therefore,

$$W_2 = \tfrac{1}{2}L_2I_2^2 \qquad [14·43]$$

During the growth of i_2, however, an emf is induced in coil N_1:

$$e_{21} = N_1 \frac{d\phi_{21}}{dt}$$

$$= N_1 \frac{d(\mathcal{P}_{21}\mathcal{F}_2)}{dt} = N_1 N_2 \mathcal{P}_{21} \frac{di_2}{dt}$$

or

$$e_{21} = L_{21} \frac{di_2}{dt} \qquad [14 \cdot 44]$$

Because N_1 is carrying current I_1 during the existence of this emf, it must receive or deliver energy as follows.

$$W_{21} = \int p\,dt = \int e_{21} I_1\,dt = \int L_{21} \frac{di_2}{dt} I_1\,dt$$

$$= L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2 \qquad [14 \cdot 45]$$

This energy is said to be stored in the mutual inductance. It must be delivered *to* the coil, by the electric circuit because, as shown in Fig. 14·9, the polarity of e_{21} with respect to the direction of I_1 is the same as e_1 which accompanied W_1. The total field energy, then, is

$$W_t = W_1 + W_2 + W_{21} \qquad [14 \cdot 46]$$

Now observe that reversal of the sequence of operations, to establish first I_2 and then I_1, would involve an emf e_{12} during growth of i_1. The energy delivered to N_2 during this current growth would be

$$W_{12} = L_{12} I_1 I_2 \qquad [14 \cdot 47]$$

Since either sequence must result in the same *total* energy stored in the magnetic fields of the two coils, and since W_1, as well as W_2, is the same by either sequence, it follows that the mutual energy must also be the same by either sequence, and we write

$$W_{12} = W_{21} \qquad [14 \cdot 48]$$

Substituting equations 14·45 and 14·47 gives

$$L_{12} = L_{21} \qquad [14 \cdot 49]$$

Further, substituting equations 14·33 and 14·39 gives

$$\mathcal{P}_{12} = \mathcal{P}_{21} \qquad [14 \cdot 50]$$

In consequence of these equations we shall use only the single subscript $_M$ for mutual inductance L_M and similarly refer to either permeance merely as the *permeance of the mutual flux path* $\mathcal{P}_M$.

14·15. Variable Mutual Inductance. For ferromagnetic materials, where μ is not constant, the foregoing discussion requires more drastic modification than was required for self-inductance. Because of possible dissymmetry in the magnetic characteristics of these materials equations 14·47, 14·48, and 14·49 are not necessarily true, and we shall consider here only the case for *un*varying μ.

14·16. Coupling Coefficient. The inductances L_1 and L_2 of two circuit elements, such as the coils N_1 and N_2 of our present study, are independent properties of the respective coils, regardless of their mutual inductance, so long as the magnetic medium is of *constant* permeability. Mutual inductance L_M, however, depends not only upon both L_1 and L_2 but also upon the shape and relative position (the geometric situation) of the two circuit elements.

It is convenient to visualize the geometric factor in terms of coupling; ***loose coupling*** implies a geometric situation permitting relatively little mutual flux while ***close coupling*** implies a geometric situation permitting relatively large mutual flux. ***Perfect coupling*** designates the idealistic limit where all flux is mutual, there being no nonmutual flux. The mutual inductance for this limit is readily computed in terms of L_1 and L_2 and, being the idealistic or limiting *maximum* value of L_M, we accordingly shall identify it by notation $(L_M)_{\max}$. The utility of the coupling concept is enhanced by providing for it a numerical significance; it is the ratio of the actual mutual inductance L_M to the limiting value $(L_M)_{\max}$.

$$K = \frac{\pm L_M}{(L_M)_{\max}} \qquad [14\cdot51]$$

The numeric K is known as the ***coupling coefficient*** and may have values ranging from zero to any value short of the $\pm$ unity limit.

It is useful to develop relations for K in terms of the *mutual* and *nonmutual inductances*. For this we return to equations 14·34 and 14·40. Substituting L_M for L_{12} and for L_{21} we have

$$L_1 = L_{11} + \frac{N_1}{N_2} L_M \qquad [14\cdot34']$$

$$L_2 = L_{22} + \frac{N_2}{N_1} L_M \qquad [14\cdot40']$$

Solving each of these equations for L_M and multiplying them together gives

$$L_M^2 = (L_1 - L_{11})(L_2 - L_{22}) \qquad [14\cdot52]$$

To obtain $(L_M)_{\max}$ for perfect coupling, we must eliminate the non-mutual flux paths $\mathcal{P}_{11}$ and $\mathcal{P}_{22}$, making $L_{11} = 0 = L_{22}$. Equation 14·52 then gives

$$(L_M)^2_{\max} = L_1 L_2$$

or

$$(L_M)_{\max} = \pm\sqrt{L_1 L_2} \qquad [14\cdot 53]$$

Expressed in words, the ***mutual inductance for unity coupling is equal to the geometric mean of the individual inductances concerned.***

The coefficient of coupling may now be expressed by

$$K = \frac{L_M}{(L_M)_{\max}} = \pm\frac{L_M}{\sqrt{L_1 L_2}} \qquad [14\cdot 54]$$

It is sometimes useful to express the coupling coefficient in terms of the *component fluxes.* This is readily done from equation 14·54 by substituting the proper $N\phi/i$ for each inductance. It is important to note that there are *two* relations for L_M.

$$L_{12} = \frac{N_2\phi_{12}}{i_1} \quad \text{and} \quad L_{21} = \frac{N_1\phi_{21}}{i_2}$$

These are *both* incorporated in the substitution by remembering that $L_M = L_{12} = L_{21}$ and by writing

$$L_M^2 = L_{12}L_{21}$$

$$K^2 = \frac{L_M^2}{L_1L_2} = \frac{\left(\dfrac{N_2\phi_{12}}{i_1}\,\dfrac{N_1\phi_{21}}{i_2}\right)}{\left(\dfrac{N_1\phi_1}{i_1}\,\dfrac{N_2\phi_2}{i_2}\right)} = \frac{\phi_{12}\phi_{21}}{\phi_1\phi_2}$$

or

$$K = \pm\sqrt{\frac{\phi_{12}\phi_{21}}{\phi_1\phi_2}} \qquad [14\cdot 55]$$

This may be expanded into the form

$$K = \pm\sqrt{\frac{\phi_{12}\phi_{21}}{(\phi_{11}+\phi_{12})(\phi_{22}+\phi_{21})}} \qquad [14\cdot 56]$$

For either form it is useful to study Figs. 14·7, 14·8, and 14·9, and observe that K is the geometric mean of ϕ_{12}/ϕ_1 and ϕ_{21}/ϕ_2 each of which represents, for its respective coil the fraction of the total flux produced by itself which is mutual. Some care seems advisable to note that ϕ_M,

the *total* mutual flux (equation 14·41), is *not* concerned in these relations for coupling coefficient.

It is also possible to show that $K \neq f(N_1, N_2)$ because

$$K = \frac{\pm \mathcal{P}_M}{\sqrt{\mathcal{P}_1 \mathcal{P}_2}} \qquad [14\cdot 57]$$

14·17. The Minus Sign. It remains to determine whether the $\pm$ which appears in equation 14·51 is physically valid or whether the minus sign must be discarded. Having shown that $L_M = N_1 N_2 \mathcal{P}_M$ it appears that L_M cannot itself be minus because no physical significance to a minus sign for N_1, N_2, or $\mathcal{P}_M$ is recognized.

Returning to Fig. 14·9 it will be observed that the direction of currents i_1 and i_2 was so assumed that the mmf's of the two coils would produce mutual fluxes ϕ_{12} and ϕ_{21} in the same direction. If one of these currents is actually in the opposite direction, the superposition of these fluxes will be arithmetically subtractive instead of additive and the energy W_{21} of equation 14·45 will be subtracted instead of added in the total expressed by equation 14·46. This is readily accounted for in the usual assignment of algebraic sense to the currents I_1 and I_2 in equation 14·45 according to their direction. The equation for total energy might be rewritten:

$$W_t = \tfrac{1}{2}L_1(\pm I_1)^2 + \tfrac{1}{2}L_2(\pm I_2)^2 + L_M(\pm I_1)(\pm I_2) \qquad [14\cdot 58]$$

It has become the *practice*, however, to assign to the mutual inductance the responsibility for indicating whether the mmf's are directing flux in the same or opposite directions, i.e., "aiding" or "bucking." The equation then recognizes only the *numerical* values of I_1 and I_2 as follows.

$$W_t = \tfrac{1}{2}L_1[I_1]^2 + \tfrac{1}{2}L_2[I_2]^2 \pm L_M[I_1][I_2] \qquad [14\cdot 59]$$

In this way we may now see that it is possible to write

$$L_M = \pm N_1 N_2 \mathcal{P}_M \qquad [14\cdot 60]$$

That is, we arbitrarily assign $\pm$ to L_M as a means for denoting *aiding* or *bucking* without violating the inherent *non*algebraic sense of N_1, N_2, or $\mathcal{P}_M$.

This concept carries through to include the coupling coefficient of equations 14·54 and 14·55 so that we refer to ***positive** and **negative** **coupling*** as meaning aiding and bucking mmf respectively.

14·18. The Variometer. When the two coils of Fig. 14·9 are connected in series, with provision for adjusting the coupling, as suggested in Fig. 14·10, one form of an adjustable inductance results.

The total inductance of the circuit is best found from the basically definitive relation

$$L = \frac{e}{di/dt} \quad [14\cdot5]$$

The emf e is comprised of four components, the self-induced and mutually induced emf's for each coil, as follows.

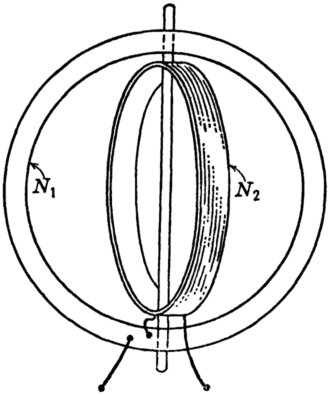


FIG. 14·10. A variometer.

For coil N_1

$$e_1 = L_1 \frac{di_1}{dt} + L_M \frac{di_2}{dt} \quad [14\cdot61]$$

For coil N_2

$$e_2 = L_2 \frac{di_2}{dt} + L_M \frac{di_1}{dt} \quad [14\cdot62]$$

The total for the series-connected coils is

$$e = e_1 + e_2 = L_1 \frac{di_1}{dt} + L_M \frac{di_2}{dt} + L_2 \frac{di_2}{dt} + L_M \frac{di_1}{dt} \quad [14\cdot63]$$

Since, for the series connection,

$$i_1 = i_2 = i \quad [14\cdot64]$$

Equation 14·63 reduces to

$$e = (L_1 + L_2 + 2L_M) \frac{di}{dt} \quad [14\cdot65]$$

And, from equations 14·5 and 14·65,

$$L = L_1 + L_2 + 2L_M \quad [14\cdot66]$$

Substituting coupling coefficient K of equation 14·54 for L_M,

$$L = L_1 + L_2 + 2K\sqrt{L_1L_2} \quad [14\cdot67]$$

This apparatus is frequently found in a form known as a ***variometer*** which is constructed (Fig. 14·10) with one coil inside of the other and on a shaft so that the coupling can be varied almost from $K = +1$ to $K = -1$. When the coils are made as much alike as possible, so that $L_1 = L_2$, approximately, the inductance is

$$L = L_1 + L_1 \pm 2K\sqrt{L_1L_1} = 2(L_1 \pm KL_1) = 2L_1(1 \pm K) \qquad [14\cdot68]$$

or

$$4L_1 > L > 0$$

14·19. Parallel-Connected Coupled Coils. When the coils N_1 and N_2 are connected in parallel as in Fig. 14·11 the results are quite

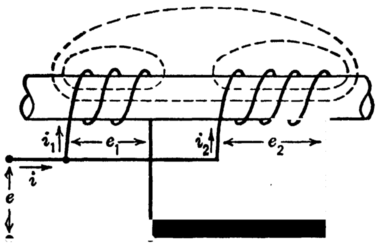


FIG. 14·11.

different from what the student might expect in the light of previous study. The basic equations are:

$$i = i_1 + i_2 \qquad [14\cdot69]$$

$$e_1 = +L_1\frac{di_1}{dt} + L_M\frac{di_2}{dt} \qquad [14\cdot70]$$

$$e_2 = +L_2\frac{di_2}{dt} + L_M\frac{di_1}{dt} \qquad [14\cdot71]$$

$$e = e_1 = e_2 \qquad [14\cdot72]$$

Solving these equations for $L = \dfrac{e}{di/dt}$, where $di/dt = (di_1/dt) + (di_2/dt)$, we find

$$L = \frac{L_1L_2 - L_M^2}{L_1 + L_2 - 2L_M} \qquad [14\cdot73]$$

Now, by introducing the coupling coefficient K from equation 14·54 in place of L_M,

$$L = \frac{L_1L_2 - K^2L_1L_2}{L_1 + L_2 - 2K\sqrt{(L_1L_2)}} = \frac{L_1L_2(1 - K^2)}{L_1 + L_2 - 2K\sqrt{(L_1L_2)}} \qquad [14\cdot74]$$

Then let us consider some limiting values of K as follows.

Case A. For $K = 0$, substituting in equation 14·74 gives

$$L = \frac{L_1 L_2}{L_1 + L_2} = \frac{1}{1/L_1 + 1/L_2} \qquad [14\cdot75]$$

If also $L_1 = L_2$, then

$$L = \tfrac{1}{2}L_1 \qquad [14\cdot76]$$

With no coupling and therefore no mutual inductance, L_1 and L_2 are as independent as though they were resistances and the well-known *reciprocal of the sum of reciprocals* relation associated with parallel connections is observed to apply.

Case B. For $K = +1$, substituting in equation 14·74 gives

$$L = \frac{0}{L_1 + L_2 - 2\sqrt{(L_1 L_2)}} = \frac{0}{(\sqrt{L_1} - \sqrt{L_2}\,)^2} \qquad [14\cdot77]$$

If $L_1 = L_2$, equation 14·77 gives

$$L = \frac{0}{0} = ? \qquad [14\cdot78]$$

Although the mathematical result is not helpful,* the physical significance is clear from Fig. 14·12*a*. Unity coupling means that the

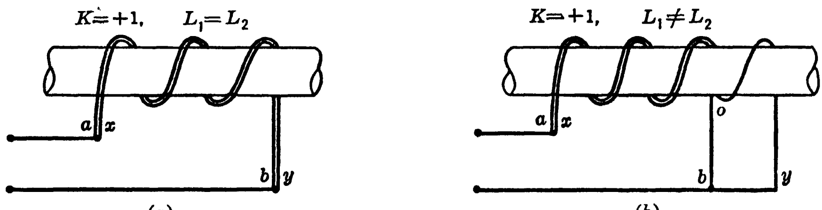


FIG. 14·12.

turns of N_1 and N_2 are wound so close to each other, turn for turn, that in the ideal case they would occupy the same space. Because this is impossible, $K = 1$ can only be approximated in reality, as shown in Fig. 14·12*a*. With $N_1 = N_2$, as required by $L_1 = L_2$, the coils in effect

* In this instance $L = 0/0$ can be avoided by returning to equation 14·74 and substituting $L_1 = L_2$ *before* substituting $K = +1$.

$$L = \frac{L_1^2(1 - K^2)}{2L_1 - 2KL_1} = \frac{L_1^2(1 + K)(1 - K)}{2L_1(1 - K)}$$

$$= \tfrac{1}{2}(1 + K)L_1 = \tfrac{1}{2}(1 + 1)L_1 = L_1$$

constitute merely one coil of $N_1 = N_2$ turns of two-strand conductor and will have the same inductance as either coil alone.

$$L = L_1 = L_2 \qquad [14 \cdot 79]$$

When $L_1 \neq L_2$ and therefore $N_1 \neq N_2$, the physical situation may be illustrated for unity coupling as in Fig. 14·12*b*.

With unity coupling, all coil turns link all flux, and any emf induced in any one turn must be identical to that in each and every other turn. For Fig. 14·12*a* where $N_1 = N_2$ and resistances are neglected, the ΣE around *baxyb* is zero, but for Fig. 14·12*b* the ΣE around *baxoyb* can be zero only when each turn emf is zero. This requirement forbids any change of flux and prevents its growth entirely in the idealized case. Yet how can the coil turns carry current without creating flux? To meet these requirements, and yet carry current through each coil in the same direction, necessitates superposing a circulating current around the loop *baxoyb* with such direction and magnitude as will make the algebraic total of all ampere-turns zero. Because $N_1 \neq N_2$ a net current in the desired direction remains possible. The paralleled, unequal, and resistanceless inductances then carry current with complete absence of linking flux and exhibit *zero total inductance* as shown by equation 14·77.

Case C. For $K = -1$, substituting in equation 14·74 gives

$$L = \frac{0}{L_1 + L_2 + 2\sqrt{(L_1 L_2)}} = \frac{0}{(\sqrt{L_1} + \sqrt{L_2}\,)^2} = 0 \qquad [14 \cdot 80]$$

Unity coupling, with the minus sign, means that the coil connections of Fig. 14·12 are reversed for one coil as shown in Fig. 14·13*a*. The

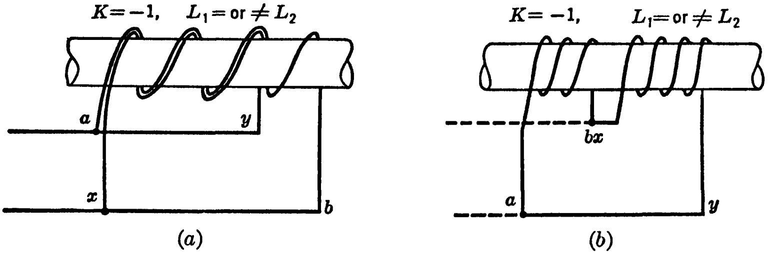


FIG. 14·13.

conditions are now best represented as in Fig. 14·13*b* which is merely a redrawing of Fig. 14·13*a*. Keeping in mind that all coil turns are close coupled so that all flux must link all turns, Fig. 14·13*b* clearly indicates that the coils are short-circuited by the loop *abxya*. By the same reasoning as for case *B* no flux can be created in the shorted turns and the

inductance, therefore, must be zero. This of course is independent of the relative magnitude of L_1 and L_2 so that $L_1 = L_2$ provides no exception in this case.

Again let it be clear that these are *ideal* conditions in that both true unity coupling and actual zero resistance can only be approached and not realized in fact. The information obtained by assuming these ideal conditions, however, is by no means inapplicable to the practical case; the ideal is commonly approximated by the actual. When resistance is not negligible it usually is not difficult to include it in an analysis similar to the above. When the permeances are not constant, however, and the validity of approximate methods is questionable, the computation readily transcends the realm of ordinary mathematical methods.

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QUESTIONS

14·1. Show by mathematical derivation that inductance is a circuit *property*.

14·2. Upon what restricted specifications is the derivation of L based? Explain what difficulty is encountered in using the term inductance for *iron* magnetic circuits.

14·3. Cite two pairs of analogous mathematical relations which justify the term *electrical mass* for inductance.

14·4. What relation exists between the polarity of a *self-induced* emf and the current in a circuit? Illustrate by sketch of circuit.

14·5. Will or can a length of *isolated motionless* conductor have (*a*) inductance, (*b*) self-induced emf when it carries:

(1) No current (at a given instant)?
(2) Constant current?
(3) Changing current?

14·6. Under what circumstances does inductance function in an electric circuit as:

(*a*) A load?
(*b*) A source (of energy)?

14·7. One construction commonly used for resistors in various instruments where the inductance must be very small is to wind the resistance on an insulating card instead of a spool. Explain how this is effective.

14·8. It is sometimes shown that from $L = \lambda/I$ we may write $LI = \lambda$ and take a derivative with respect to t to give $L(di/dt) = d\lambda/dt$. Explain why this is fallacious when *differential* inductance is involved.

14·9. Refer to equation 14·1 and explain how differential permeability and differential permeance circumvent the requirement that permeance be constant to obtain equations 14·3 and 14·4.

14·10. Explain why the energy stored in the magnetic field of an inductance is not $W = \frac{1}{2}Li^2$ for ferromagnetic core materials.

14·11. Explain why the concept of mutual inductance depends upon the permeance being constant.

14·12. Explain why $e = L_M(di/dt)$ is *not* a reversible relation like $e = L(di/dt)$.

14·13. Explain how the minus sign has meaning for mutual inductance and not for self-inductance.

14·14. Draw an appropriate circuit of two coils in series and with plus mutual inductance. Show all polarities and direction of emf and current.

14·15. Explain what is meant by *coupling* and define the *coupling coefficient.*

14·16. Devise another approach than that in the text to show that the energy stored in a mutual inductance is $W = L_M I_1 I_2$.

14·17. Derive equation 14·57 and show of what quantities it represents the geometric mean.

14·18. What would $K = -1$ signify for two coupled coils and why is it a practical impossibility?

14·19. Sketch your concept of a variometer and show positions for maximum and for minimum inductance.

14·20. For some purposes the inductance of a tubular resistor or rheostat made by winding wire in a helix on a ceramic tube is objectionable. One construction for minimizing this inductance consists in first doubling the wire back on itself (U-shaped) before winding, so that both ends of the wire are at the same end of the tube. Prove mathematically that the inductance approaches zero.

14·21. Given two coils connected in parallel, with coupling $K = \pm 1$, develop the mathematical relation for the total inductance when $N_1 = N_2$.

14·22. Derive equation 14·73, given equations 14·69–14·72.

14·23. If the coils $N_1 < N_2$ of Fig. 14·12*b* are carrying a growing total current i from common terminal *ax* to *by*, compute the *actual* current in each coil, $i_1 = f(N_1,N_2,i)$ and $i_2 = f(N_1,N_2,i)$, and show the direction of each together with i on a sketch.

PROBLEMS

14·1. A wooden ring having a mean magnetic path length of 120 cm and a cross section of 25 sq cm, is uniformly wound with a coil having $N_1 = 1000$ turns of wire. Closely wound over this coil is a second coil of $N_2 = 4000$ turns. The terminals of each coil are brought out separately.

(*a*) What is the inductance L_1 of the first coil alone?

(*b*) What is the inductance L_2 of the second coil alone?

(*c*) What relation is there between L and N?

(*d*) If the two coils are connected electrically in series so that their mmf's are additive, what will be the inductance of the total winding?

14·2. For the structure concerned with Problem 12·1 compute the differential inductance L_d of coil A when it alone is connected to:

(*a*) 110 volts.

(*b*) 55 volts.

(*c*) 27.5 volts.

14·3. A coil with closed sheet steel core of uniform sectional area is to have an incremental inductance of 10 henrys between 15 and 25 ma. The highest flux density to be permitted is 17 kilogauss and a sectional area of 3 sq in. is contemplated.

(*a*) Compute the number of turns required.

(b) Compute the length of magnetic circuit.

(c) Are the results in (a) and (b) reasonable? If not, what changes in design should be made?

14·4. (a) Compute the differential inductance of coil B in Problem 12·1 when it is carrying such current as will produce the same flux that A produces on 110 volts.

(b) How is this value related to Problem 14·2a?

14·5. A wooden circular ring is uniformly wound with a coil of 800 turns of fine wire. The inductance is 1×10^{-3} henrys. Another coil of 1200 turns is wound uniformly over the first, the area bounded by the turns being practically the same as for the first coil.

(a) What is the permeance of the core?

(b) What is the inductance of the second coil?

(c) What is the mutual inductance between the coils?

(d) What is the total inductance of the two coils when they are connected in series so that their turns trace around the core in (1) the same direction? (2) the opposite direction?

(e) What voltage will be induced in the coil of 2000 turns if the current throughout it increases at the rate of 600 amp per second?

(f) When the current becomes constant, at 3 amp, how much kinetic energy is associated with the magnetic field of the coil of 2000 turns?

14·6. Two coils A and B of 100 and 200 turns, respectively, in air have nonmutual flux $\phi_{AA} = 50$ kilolines and $\phi_{BB} = 80$ kilolines when connected in series and carrying 10 amp. Their coupling coefficient is $K = 0.6$.

(a) Compute the mutual flux.

(b) Compute the total inductance of the coils connected in series aiding.

(c) What emf will be induced in one coil when current in the other is changing 100 amp per second?

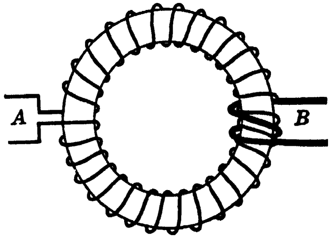
14·7. Two air-core coils have five times as much inductance when connected in series aiding as for series opposing. When each is tested individually coil A has 50 per cent more turns and twice the inductance of B. Compute:

(a) The coupling coefficient.

(b) The ratio of the nonmutual inductances L_{AA}/L_{BB}.

(c) The ratio of the nonmutual fluxes ϕ_{AA}/ϕ_{BB}.

14·8. Coil A of 1000 turns is wound in the form of a uniformly distributed air-core toroid as shown in the accompanying figure. The magnetic flux path has a 10-cm length and a 5-sq cm section. Wound over a *small portion* of coil A is a concentrated coil B which has self-inductance $L_B = 80$ microhenrys.



An emf of 4 volts is produced at the terminals of A when current in B is changed at the rate of 10,000 amp per second.

(a) Compute the value of inductance L_A.

(b) Compute the coupling coefficient of the two coils.

(c) What per cent of all flux ϕ_A produced by a current in A links B?

(d) What per cent of all flux ϕ_B produced by a current in B links A?

(e) How many turns has coil B?

CHAPTER XV

CIRCUITS WITH RESISTANCE AND INDUCTANCE

15·1. Resistance and Inductance in Series. In general, electrical conductors and electrical apparatus possess both inductance and resistance intimately combined in the same structure. The series connection of R and L is important not alone because we may actually connect resistors and inductances in series but also because the *inherent combination* of R and L parameters in electrical conductors and apparatus is

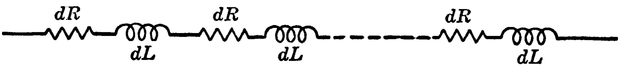


FIG. 15·1.

commonly the equivalent of a *series connection.* This is evident because the voltage of self-inductance $e = L(di/dt)$ is found to *add* to the ohmic drop $e = Ri$ as though in series, and because each voltage is related to the *entire* current i instead of to only part of it as behavior in parallel would require. The circuit for a simple conductor comprises an infinite number of elements dR and dL as indicated in Fig. 15·1. When these are the only factors present in the analysis, the dR and dL elements integrate without regard for their alternate sequence, and function as though lumped into separately integrated entities as in Fig. 15·2. The ratio of inductance to resistance varies over a wide range according to the nature and design of the particular equipment.

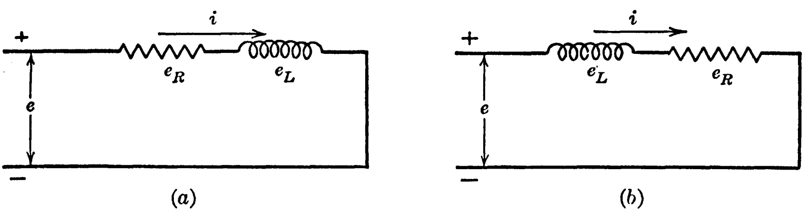


FIG. 15·2.

Any attempt to distinguish between the circuit of Fig. 15·2*a* and that of Fig. 15·2*b*, on the basis that the current must reach R before L in Fig. 15·2*a* and that it must reach L before R in Fig. 15·2*b is wholly erroneous. To avoid such illusions* it is imperative that the physical

picture of current flow in a conductor presented in Chapter II be clearly in mind. Within the scope of our study here, there is no concept of a pulse or wave of current propagated around the circuit; the same flow of current at each and every instant exists in each and every section of the simple series circuit in accord with Kirchhoff's law.

From fundamental relations an equation for the total emf is easily obtained as follows.

$$e_R = Ri, \qquad e_L = L\frac{di}{dt}$$

$$e = e_R + e_L = Ri + L\frac{di}{dt} \qquad [15\cdot1]$$

Solution of this equation of course requires knowledge of e as a function of i or t, and is easy only for a few special cases which, fortunately, are of practical importance. We shall consider here only *constant* applied voltage $e = E$.

15·2. Series R and L with Constant E. Let the circuit be as in Fig. 15·3 so that constant voltage E is applied when switch S is snapped closed. Equation 15·1 becomes

$$E = Ri + L\frac{di}{dt} \qquad [15\cdot2]$$

At the initial instant $t = 0$, representing closure of switch S, the current is zero and consequently Ri must be zero. It follows that at this

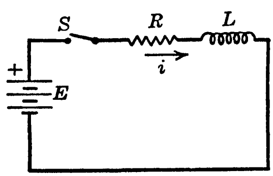


FIG. 15·3.

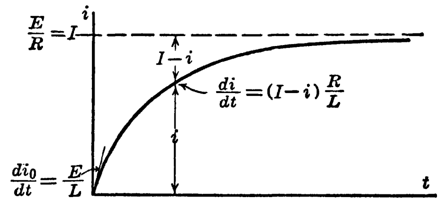


FIG. 15·4.

instant the entire voltage E must appear across the *inductance* in accord with equation 15·2 so that

$$E = 0 + L\frac{di_0}{dt} \qquad [15\cdot3]$$

This means, of course, that R has nothing to do with the *initial* rate of current change di_0/dt; it is determined entirely by E and L.

$$\frac{di_0}{dt} = \frac{E}{L} \qquad [15\cdot4]$$

As the current grows, more and more of voltage E will be required for the resistance drop Ri and less and less will be available for the inductance to produce current growth di/dt. Eventually the entire voltage E will be required for Ri, and growth di/dt must cease because there is no voltage left for the inductance.

The final value of current I is therefore determined entirely by E and R, independent of L, and equation 15·2 becomes simply

$$E = RI + 0$$

or

$$I = \frac{E}{R} \qquad [15\cdot5]$$

Remembering that inductance is an inertia-like quantity, that resistance is a friction-like quantity, and that voltage and current are somewhat analogous to force and velocity, respectively, it is sometimes helpful to observe that the application of a constant voltage to series R and L produces current in much the same way that the application of a constant force to a mass with dynamic friction produces velocity. If we apply constant force to propel a boat from standstill, all the force is at first available for acceleration and none is required for friction. As the boat acquires velocity, more and more of the force is required for friction and less and less is available for acceleration until all the force eventually is required for friction and there is no further acceleration. In fact the same equation (15·2) applies to the mechanical case when the frictional force is proportional to the velocity.

Let us now study the manner in which the growth of current (acceleration of electrons) proceeds from the initial to the final status. For this purpose we will first arrange equation 15·2 to give

$$\frac{di}{dt} = \frac{E}{L} - \frac{R}{L}i$$

Substituting equation 15·4 we have

$$\frac{di}{dt} = \frac{di_0}{dt} - \frac{R}{L}i \qquad [15\cdot6]$$

This indicates that the rate of current growth *at any instant* is equal to the initial rate reduced by an amount directly proportional to the current at that instant and to the ratio R/L. The larger R/L is, the more rapidly will growing i choke off its own growth.

Equation 15·2 may also be arranged to give

$$\frac{di}{dt} = \left(\frac{E}{R} - i\right)\frac{R}{L}$$

or substituting equation 15·5,

$$\frac{di}{dt} = (I - i)\frac{R}{L} \qquad [15\cdot7]$$

This indicates (Fig. 15·4) that ***the rate of current change at each instant is directly proportional to the amount (I − i) by which the change is yet incomplete, and to the ratio R/L.***

This is characteristic of many phenomena in mechanical, hydraulic, and thermal as well as electrical engineering. It is aptly typified by the behavior of a small boy approaching a distasteful task—the nearer he gets the slower he moves. With current, however, we must be clear that it is the *growth* of current and not the current itself which slows up as the objective is approached.

Mathematically this characteristic is represented by an exponential function. Because equation 15·7 is a simple well-known type of differential equation, the exponential is readily disclosed by solving for $i = f(t)$. Direct integration will suffice.

For this purpose equation 15·7 is arranged to give

$$\frac{di}{I - i} = \frac{R}{L}\,dt$$

Integrating

$$-\ln(I - i) = \frac{R}{L}t - \ln K$$

$$I - i = K\epsilon^{-\frac{R}{L}t}$$

$$i = I - K\epsilon^{-\frac{R}{L}t} \qquad [15\cdot8]$$

Substituting the initial conditions that $i = 0$ when $t = 0$, we evaluate integration constant K:

$$0 = I - K\epsilon^{-0}$$

$$K = I$$

and the solution becomes

$$i = I - I\epsilon^{-\frac{R}{L}t} \qquad [15\cdot9]$$

The exponential term $I\epsilon^{-(R/L)t}$ is evidently a decaying function which approaches zero asymptotically as time approaches infinity. Because,

at $t = 0$, $\epsilon^{-(R/L)t}$ is unity, its coefficient or *amplitude* I represents the initial value of the term. For current *growth* we wish to approach value I rather than zero, which requires that I appear in $i = f(t)$ as a constant term to which the exponential term is "added" with a *minus* sign representing *growth* rather than decay.

Thus we find that $i = f(t)$ comprises two terms:

1. The final or ***steady-state*** value, as it is usually called, which is the objective of the change.
2. A temporary or ***transient*** term which provides for the transition from initial to final value by subtraction from the *steady-state* term.

15·3. The Decay of Current in the Series *RL* Circuit. Let us now consider how the current, which we have grown to maturity or *steady state* in the *RL* circuit, is brought back to zero again. For this purpose we arrange the apparatus as in Fig. 15·5, so that E may be removed

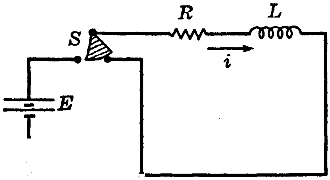


Fig. 15·5.

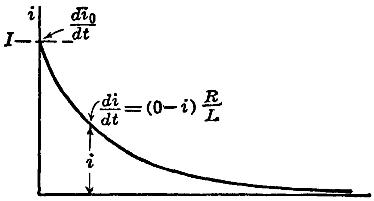


Fig. 15·6.

without opening the *RL* circuit. It is essential that the removal of voltage *not* be permitted also to remove the *path* by which the current may continue to "coast to rest." Merely to open the switch of Fig. 15·3 would be like suddenly ramming the previously discussed boat into a dock instead of letting it coast to rest. Switch S in Fig. 15·5, while momentarily short-circuiting the voltage source, is designed to remove E quickly without opening the *RL* circuit.

The Kirchhoff equation for Fig. 15·5 is the same as equation 15·2 except that $E = 0$.

$$0 = Ri + L\frac{di}{dt} \qquad [15·10]$$

At the initial instant $t = 0$, representing operation of switch S to the position shown, the current of course is the final value I to which it had grown according to equation 15·5. The initial rate of current change is then obtained by substituting this value in equation 15·10.

$$\frac{di_0}{dt} = -\frac{RI}{L} \qquad [15·11]$$

Because, for growth, $RI = E$, we may write for comparison,

$$\frac{di_0}{dt} = -\frac{E}{L} \qquad [15\cdot12]$$

Comparing equation 15·12 with equation 15·4 we find that the initial rates of *decay* and *growth* have identical magnitudes.

The rate of decay at any instant, of course, is (from equation 15·10)

$$\frac{di}{dt} = -\frac{R}{L}i \qquad [15\cdot13]$$

Because i now represents at each instant the amount by which the change (to zero) is yet *incomplete* we find that the process of decay is governed by the same law which, following equation 15·7, we established for growth. The graph is given in Fig. 15·6.

Let us now solve equation 15·13 for $i = f(t)$. Rearranging and integrating,

$$\frac{di}{i} = -\frac{R}{L}dt$$

$$\ln i = -\frac{R}{L}t + \ln K$$

$$i = K\epsilon^{-\frac{R}{L}t} \qquad [15\cdot14]$$

Substituting the initial condition that $i = I$ when $t = 0$, to find K

$$I = K\epsilon^{-0}$$

$$K = I$$

and the solution is

$$i = I\epsilon^{-\frac{R}{L}t} \qquad [15\cdot15]$$

Because we found, in analyzing the growth equation, that the exponential or *transient* term is inherently a *decaying* function, it is but to be expected that $i = f(t)$ for decaying current *should* comprise *only* the exponential term. Equation 15·15 confirms this reasoning and contains the initial value of current as the amplitude of the function.

15·4. The Time Constant. In each of the foregoing cases it is impossible to compute the time required for the change of current from initial to final value because it is mathematically infinite. Nevertheless, to engineering accuracy, the actual time required for the current to approximate its final value seldom exceeds a few seconds, and is more commonly a matter of milliseconds or even microseconds. Evidently some

practical measure or index of the time factor in these transient phenomena is necessary. This is commonly provided by what is called the *time constant* of the circuit which may be defined by either of two concepts, as follows.

First let us observe that, because there is an invariable law (exponential) by which the current change proceeds after its initiation, the *initial* rate of growth di_0/dt is an index of the *overall* speed with which the change will take place. This may be likened to the shooting of a bullet vertically into the air; the muzzle velocity of the bullet is an index of its overall speed because, beyond the muzzle, the action of friction and gravity are invariable functions of the muzzle velocity. To measure *how long,* we must know *how far* as well as *how fast,* and our time index must contain the amount of the change. Thus we conceive

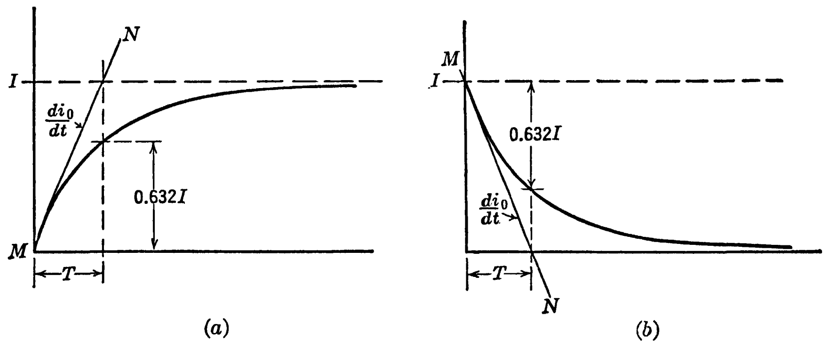


FIG. 15·7.

that ***the time constant is the time which would be required to execute the complete current change entirely at the initial rate.*** The linear current change of this concept is represented in Fig. 15·7 by MN which, drawn tangent to the actual current curve at $t = 0$, has slope di_0/dt.

If we let T represent the time constant, its value is as dimensioned on Fig. 15·7, and we may write

$$\frac{di_0}{dt} = \frac{I}{T} \qquad [15\cdot16]$$

But we have previously found (equations 15·4, 15·11) that the magnitude of di_0/dt is

$$\frac{di_0}{dt} = \frac{E}{L} = \frac{RI}{L} \qquad [15\cdot17]$$

It follows that

$$T = \frac{L}{R} \qquad [15\cdot18]$$

Thus we find that the time constant depends only on the physical composition of the circuit and is a *property* of the circuit as well as an *index* of its transient performance.

Returning now, to develop the second concept of the time constant, we pursue the actual $i = f(t)$ curve and determine what per cent of the entire change is actually accomplished in the time of equation 15·18. This is readily found (Fig. 15·7*a*) by substituting this value of time in equation 15·9 to obtain the ratio i/I.

$$\frac{i}{I} = 1 - \epsilon^{-\left(\frac{R}{L}\right)\left(\frac{L}{R}\right)} = 1 - \epsilon^{-1} = 1 - \frac{1}{2.718} = 0.632 = 63.2\%$$

The same result is obtained from equation 15·15 for decay if care is exercised to observe that the amount of change is then $I - i$ (Fig. 15·7*b*), so that the per cent change achieved in time T is

$$\frac{I - i}{I} = 1 - \frac{i}{I} = 1 - \epsilon^{-\left(\frac{R}{L}\right)\left(\frac{L}{R}\right)} = 1 - \epsilon^{-1} = 0.632 = 63.2\%$$

Thus we find that ***the time constant represents the time required for the current to achieve 63.2 per cent (approximately two-thirds) of its change from initial to final value.***

At this point it is well that the reader review the equations which have been developed for current growth and decay, expecially 15·6, 15·7, and 15·11, and observe how this time constant, ratio L/R, not only concerns the above two concepts but also *actually governs the time rate of the phenomena throughout their progress.* It must not be overlooked that, when $i = 0$ at $t = 0$, the initial rate of growth is actually independent of R (equation 15·4) although this is true only for the *initial* instant.

15·5. The General Case for Series *RL* with Constant *E*. The foregoing discussion of current growth and decay has been confined to the special initial conditions of $i = 0$ for growth and $i = I = E/R$ for decay. Let us now consider the case represented by Fig. 15·8 where switch S is operated at intervals of time too short even for approximate completion of the current change, as graphed in Fig. 15·9. Here also we introduce a different resistance for decay than for growth because R_1 is effective in the RL circuit only during *decay* (with S open).

For the first time-interval (0 to t_1) the previously derived equation 15·9 of course applies, although the growth only reaches value I_1 before decay is initiated.

$$i = I - I\epsilon^{-\frac{R}{L}t} \qquad [15 \cdot 19]$$

From t_1 to t_2 equation 15·15 applies except that the initial value for decay is necessarily the value I_1 reached during growth, and R is augmented by R_1.

$$i = I_1 \epsilon^{-\left(\frac{R+R_1}{L}\right)(t-t_1)} \qquad [15\cdot20]$$

From t_2 to t_3 we return to equation 15·8 and reevaluate constant K for the new initial condition that $i = I_2$ when $t - t_2 = 0$:

$$I_2 = I - K\epsilon^{-0}$$

$$K = I - I_2$$

or

$$i = I - (I - I_2)\epsilon^{-\frac{R}{L}(t-t_2)} \qquad [15\cdot21]$$

From t_3 to t_4 we repeat 15·20 except for the value of t:

$$i = I_1 \epsilon^{-\left(\frac{R+R_1}{L}\right)(t-t_3)} \qquad [15\cdot22]$$

From t_4 the procedure obviously continues with repetition of the above.

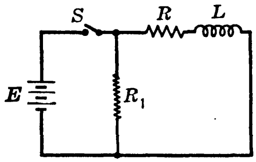


Fig. 15·8.

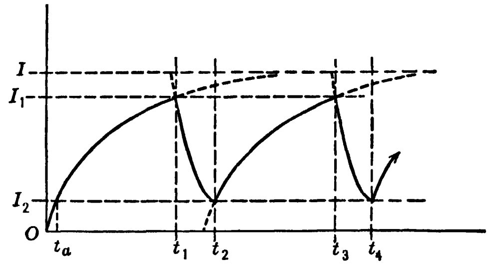


Fig. 15·9.

Summary of Solution for $i = f(t)$.

We are now in position to set up a general equation for $i = f(t)$ which may be interpreted to cover all the cases here considered. There are two limiting values of i to be clearly recognized.

(*a*) The *initial* value of the current (I_0) *from* which change occurs.

(*b*) The *final* value of the current ($I = E/R$) *to* which the change proceeds as an *objective*. Note clearly that voltage E here signifies the value *actually in the RL circuit during the current change* which, for the cases here considered, is zero during decay.

The current at any instant $i = f(t)$ comprises two terms.

1. The final or objective value I from (*b*) known as the *steady state* value.

2. The exponential *transition* term which, as in equation 15·21, must have an amplitude equal to the amount of possible change, or the differ-

ence between (*a*) and (*b*), and an exponent equal to time multiplied by the minus reciprocal of the time constant during the change.

$$(I - I_0)\epsilon^{-\frac{R}{L}t}$$

or

$$(I - I_0)\epsilon^{-\frac{t}{T}} \qquad [15 \cdot 23]$$

The complete equation then is item (1) minus item (2).

$$i = I - (I - I_0)\epsilon^{-\frac{R}{L}t} \qquad [15 \cdot 24]$$

It is helpful to keep clearly in mind as a basic concept for phenomena of exponential growth and decay that only three factors are required to describe the process at *any instant:*

1. The present value of the current (where it *is*).
2. The ultimate value or *goal* of the current (where it is *going*).
3. The time constant (how *fast* it is going).

What *has* happened in no way concerns what *will* happen; the future is independent of the past. An illustration of this is found in Fig. 15·9. Here the graph of growth of i toward value I from value I_2 is the same from t_2 to t_3 as from t_a to t_1, although what has been happening to the current just before time t_a is entirely different from what happens just before time t_2.

That the two curves, $i = f(t)$ from t_a to t and $i' = f'(t)$ from t_2 to t, are identical is readily shown by reference to equations 15·19 and 15·21 already developed for these curves. From equation 15·19 we find that I_2 is

$$I_2 = I - I\epsilon^{-\frac{R}{L}t_a} \qquad [15 \cdot 25]$$

Now with t_a as a new starting point for counting time we let $t' = t - t_a$ and equation 15·19 gives for $t = t_a + t'$:

$$i = I - I\epsilon^{-\frac{R}{L}(t_a + t')}$$

or

$$i = I - \left(I\epsilon^{-\frac{R}{L}t_a}\right)\left(\epsilon^{-\frac{R}{L}t'}\right) \qquad [15 \cdot 26]$$

Substituting equation 15·25 in 15·26 to eliminate t_a we find

$$i = I - (I - I_2)\epsilon^{-\frac{R}{L}t'}$$

or

$$i = I - (I - I_2)\epsilon^{-\frac{R}{L}(t - t_a)} \qquad [15 \cdot 27]$$

This is evidently identical to equation 15·21 and shows that the curves from t_a to t and from t_2 to t are identical in spite of the difference in events immediately preceding the respective instants t_a and t_2.

This concept is further supported by equations 15·7 and 15·13 for current change di/dt which contain only symbols for the present and future values of current and no information about the past. Equation 15·7 may be written

$$\frac{di}{dt} = \frac{I - i}{L/R} = \frac{I - i}{T} \qquad [15\cdot 28]$$

in this form the relation clearly indicates that at *any* instant the rate of current growth is such that continuation at that rate would complete the growth from present value i to final value I in the time of the time

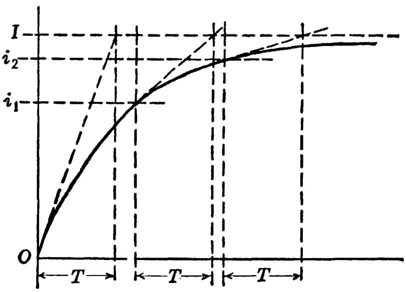


Fig. 15·10.

constant T. As shown by Fig. 15·10, this means that the defined value of T in terms of *initial* rate of current growth is but one case of a general property of the exponential function. This supports the concept that the growth is independent of the past because *any instant may be considered to represent a new start* ($t = 0$) in every sense.

15·6. The Voltage across R and L. The voltage across R, of course, is Ri, and equation 15·24 gives

$$e_R = RI - R(I - I_0)\epsilon^{-\frac{R}{L}t} \qquad [15\cdot 29]$$

By choosing the proper scale of voltage the graph of $i = f(t)$ for a fixed value of R will also serve as the graph of $e_R = f(t)$, as in Fig. 15·11.

The voltage across L is $RI - e_R$ which gives

$$e_L = RI - RI + R(I - I_0)\epsilon^{-\frac{R}{L}t}$$

or

$$e_L = R(I - I_0)\epsilon^{-\frac{R}{L}t} \qquad [15\cdot 30]$$

Figure 15·11 shows the graph for operation of switch S in the circuit of Fig. 15·5. It is to be observed clearly that emf e_L across the inductance reverses polarity immediately as the current begins to decrease, although there is, of course, no reversal of current or of the polarity of e_R. This is in accord with our study of inductance alone.

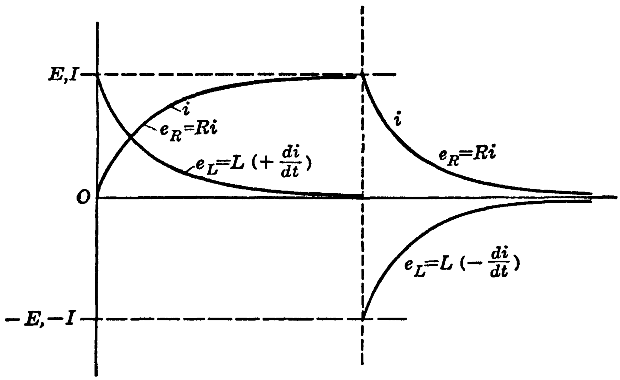


FIG. 15·11.

Referring to the circuit of Fig. 15·8 it is important to observe that the initial stage of current decay involves a higher voltage than any encountered during current growth. This is due to the increase of resistance R to value $R + R_1$. Because the inertia-like inductance prevents any immediate change or discontinuity in the current, the initial voltage for decay is $I_1(R + R_1) > I_1R$. This voltage may readily exceed the applied E and, when growth to value I is permitted before initiating decay, it *must* do so. The inductance voltage for *decay* of course is equal to the total resistance voltage so that the value of this voltage may be computed either as an e_L or an e_R according to preference or discretion.

15·7. Power and Energy for R and L. The power supplied to a series RL circuit from a constant-voltage supply is

$$p = Ei = Ri^2 + Li\frac{di}{dt} \qquad [15\cdot31]$$

The energy supplied from $i = 0$, $t = 0$, to any value i at time t is

$$W = \int p\,dt = R\int i^2\,dt + L\int i\,di$$

$$W = R\int i^2\,dt + \tfrac{1}{2}Li^2 \qquad [15\cdot32]$$

This, it will be recalled, represents the sum of the individual energies supplied to R and to L. In general the energy for L is readily found, while that for R involves integration which, although not so simple as the computation for L, is not difficult. When the total energy is desired it may be best computed by evaluating:

$$W = E \int i\,dt \qquad [15\cdot33]$$

15·8. Inductance in A-C Circuits. Because inductance makes itself felt only when i is *changing*, it is only of transient importance in d-c circuits. But in a-c circuits, where e and i are practically always in process of change, inductance is one of the star performers and merits special study in this role.

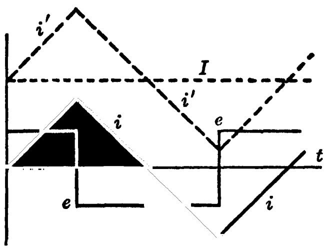


FIG. 15·12.

When a curve of $i = f(t)$ is known, it is a simple graphical procedure to find the emf that accompanies this current for the inductance. From $e = L(di/dt)$, e (at any instant) is merely L times the slope of the i-t curve as in Fig. 15·12.

When the emf $e = f(t)$ across an inductance is known, i may be found graphically by integration: $i = \frac{1}{L}\int e\,dt$. Since $\int e\,dt$ is merely the area under the e-t curve, a planimeter may be used to find i for each instant and thus to plot the i-t curve. Other processes of graphical integration are also available if desired.

Since it is only the *slope* of the current and not its actual ampere value that is concerned in $e = L(di/dt)$ it may be seen that any current such as i', which has the same shape and phase as i, will produce the same $e = f(t)$. This amounts to saying that $e = f(t)$ is not affected by the superposition of any constant component of current upon the original $i = f(t)$.

Furthermore it may be noted in the integration $i = \frac{1}{L}\int e\,dt$ that this direct current component appears in the form of the usual constant of integration which will be encountered in the actual evaluation of the integral without definite limits.

15·9. Practical Aspects of the D-C Component. The constant component of current has extensive practical significance. For a familiar illustration consider an automotive ignition coil as in Fig. 15·13. The cyclic operation of switch S, representing the *breaker points*, produces

pulsating battery current around the circuit. The alternating component of the current, by mutual induction, induces emf in the coupled high-tension coil N_2. For economy of design the magnetic circuit includes considerable iron. If the circuit were entirely of iron, the mmf due to the direct current would be able to produce enough flux to reach

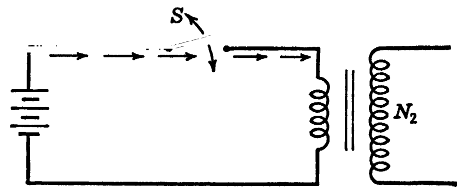


FIG. 15·13.

a density well up on the *B-H* curve. As observed earlier, the permeability of the iron is considerably reduced at these higher densities and the inductance, which depends upon the permeance of the magnetic circuit, is correspondingly reduced. To reduce this effect, it is usual to provide *air gaps* of suitable length in the magnetic circuits of these and other iron-core inductances which are to carry d-c components. Inductances of this character are common not only in induction coils but also in communication equipment, especially as components of various *filter circuits*. Most radio-receiving sets use them in the power supply filters and in the audio or output transformers.

An interesting application, which illustrates that this saturation effect is not always disadvantageous, is found in the control of stage lighting. The circuit for this so-called *reactance dimmer* is given in Fig. 15·14 which shows coils *NN* inserted in the a-c line to the lamps. The inductance of coils *NN* is controlled by the center coil which, carrying *direct* current, produces flux as shown and alters the permeance $\mathcal{P}_d$ of the magnetic circuit according to the amount of this *control* current. By suitable design, the amount of the direct current can be made relatively small so that the power loss is much smaller than it would be if the alternating current were controlled directly by *resistance dimmers*.

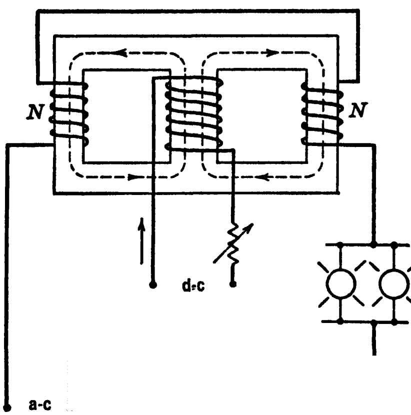


FIG. 15·14. A reactance dimmer.

The dimming effect of the inductance upon the a-c circuit will become apparent from the analysis which follows for sinusoidal current and constant (but adjustable) permeance.

15·10. Inductance for Sinusoidal Current and Voltage. When $i = I_m \sin \omega t$ through an inductance, the emf $e = L(di/dt)$ is readily found either graphically (Fig. 15·15), or analytically as follows.

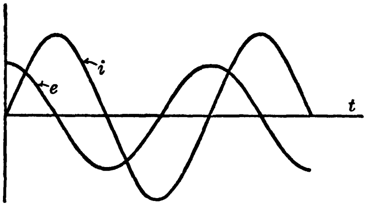


FIG. 15·15. Sinusoidal voltage and current for constant inductance.

$$e = L \frac{d}{dt} (I_m \sin \omega t) = L\omega I_m \cos \omega t$$

or

$$e = (\omega L I_m) \sin\left(\omega t + \frac{\pi}{2}\right) \qquad [15\cdot34]$$

Because the dimension of e is volts and because $\sin\left(\omega t + \frac{\pi}{2}\right)$ is dimensionless, the coefficient or *amplitude* of the sine function must have the dimension *volts* and must be the *maximum value* of the emf e. Thus, we write

$$\omega L I_m = E_m \qquad [15\cdot35]$$

and

$$e = E_m \sin\left(\omega t + \frac{\pi}{2}\right) \qquad [15\cdot36]$$

We are now in position to observe from equations 15·35 and 15·36 three important properties peculiar to an inductance carrying ***sinusoidal*** current:

1. *Sinusoidal current* through inductance is accompanied by *sinusoidal emf* across the inductance.

2. The *emf leads* the current by 90 degrees or is in leading quadrature with the current. It is equally true that the *current lags* the emf by 90°. (Fig. 15·15.)

3. The amplitude of the emf is *directly proportional to the frequency* ($\omega = 2\pi f$) as well as to the current amplitude and to the inductance.

15·11. Inductive Reactance. The dimension of (volts)/(amperes) is ohms, and it is pertinent to inquire whether equation 15·35 written

$$\frac{E_m}{I_m} = \omega L \qquad [15\cdot37]$$

may have meaning in this sense. Indeed it does; we create a new symbol:

$$X_L = \omega L \qquad [15\cdot38]$$

and call it ***inductive reactance.*** While X_L, like R, is measured in ohms it differs in at least four major respects.

1. Unless e and i are *sinusoidal,* reactance is *meaningless* and we must rely wholly on the basic concept as expressed by the fundamental $e = L(di/dt)$. It is to be noted further that the foregoing applies only to *constant* inductance. If $L = f(i)$, then $\omega L = f(i)$, and X_L is neither a useful quantity nor admissible under the restricted concept defined by the term *reactance.*
2. Reactance is a function of *time* or frequency. The specification of a *reactor* must include frequency as well as reactance (ohms) and current.
3. Reactance is defined by the ratio $X_L = E_m/I_m$ but, unlike R, is *not* also the ratio e/i where e and i are respectively the voltage across and current through the inductance at any *instant.*
4. Reactance dissipates no energy in heat.*

Items 1–3 are shown by equations 15·36 to 15·38, while item 4 will be developed presently. Some liberty is commonly taken with item 1 and the term ***effective reactance*** will be encountered when the concept of reactance is extended to wave shapes which depart from the sinusoidal. This approximation is made only at the risk of eventually encountering a situation where the results are misleading. Until some experience has been attained in dealing with such matters the approximation is dangerous. An extreme example is indicated in the current of Fig. 15·12 which unquestionably departs too much from the sine wave to warrant any attempt to associate it with the reactance concept.

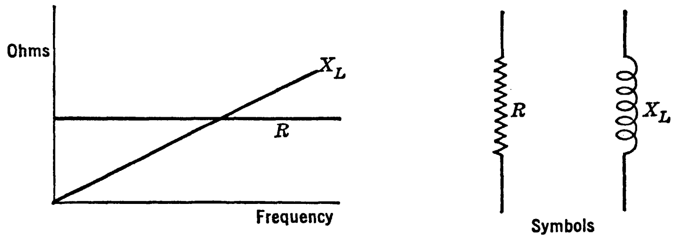


FIG. 15·16.

Item 2 is represented graphically in Fig. 15·16, and analytically by equation 15·38 which is expanded to give

$$X_L = \omega L = 2\pi f L = (2\pi L)f \qquad [15\cdot39]$$

or

$$X_L = (2\pi N^2 \mathcal{P})f \qquad [15\cdot40]$$

* This is not to be confused with the structure known as a *reactor* (Art. 15·12) which necessarily has some resistance as well as reactance, and dissipates energy as usual in the resistance.

It is highly important to recognize and retain firmly this simple but very basic frequency characteristic of inductive reactance. Much of the application of reactors is based directly upon this property: ***Inductive reactance is directly proportional to frequency.***

In the same sense that we have written for resistance the *constructional relation* $R = \rho(l/A)$ we may further expand equation 15·40 to give the corresponding constructional relation for inductive reactance.

$$X_L = 2\pi f N^2 \mu \frac{A}{l} \qquad [15\cdot 41]$$

The resistance characteristic shown in Fig. 15·16 of course assumes that the frequency is not sufficiently high to make skin effect an important factor.

15·12. Reactors. Inductive reactors (usually simply *reactors*) are a common commercial product with an extensive range of size, design, and application. Like resistors, they must be specified not only by *ohms* but also by *current-carrying capacity* or, in the larger sizes, by *kilovolt-amperes*. *Frequency*, as before observed, must be included in the specification. It is not to be construed that the specification of reactors ends with these items—many factors not to be considered here are involved in their design. Most of these are in common with *transformers* and are studied accordingly.

Air-core reactors are widely used to protect electrical apparatus from lightning disturbances and other abnormalities of current by inserting them in series with distribution and transmission lines. These reactors are designed to have negligible effect (X_L) at normal frequency but offer an effective inertia to the exceptionally abrupt changes of current during the disturbance. Air-core reactors are common in many communication devices—radio especially.

Iron-core reactors (as before mentioned) are used instead of resistors for theater dimmers (to reduce voltage of lamps), with a-c arcs for stabilization, in filters for smoothing rectified alternating current, in selective filters for communication and power-relay circuits, and in numerous other applications. The cores are usually assembled by stacking sheet steel stampings electrically insulated from one another so as to reduce the magnitude of otherwise intolerable *eddy currents* induced in the core. For communication equipment, cores of powdered iron (dust cores) have attained prominence.

When iron-cored reactors are operated at values of current, and consequent flux density, which involve material departure from the linear region of the *B-H* curve the concept of reactance is violated and its value becomes an indeterminable quantity.

15·13. Power and Energy for Reactance X_L. In general, electric power may be expressed by $p = ei$. For the inductive reactor we have found that when $i_L = I_m \sin \omega t$, $e_L = X_L I_m \cos \omega t$. It follows that $p_L = i_L e_L = (I_m \sin \omega t)(X_L I_m \cos \omega t) = X_L I_m^2 \sin \omega t \cos \omega t$
or

$$p_L = \tfrac{1}{2} X_L I_m^2 \sin 2\omega t \qquad [15\cdot42]$$

Graphically, p_L may be found by plotting the product $p = ei$ for several instants of time, or equation 15·42 may be plotted directly as in Fig. 15·17.

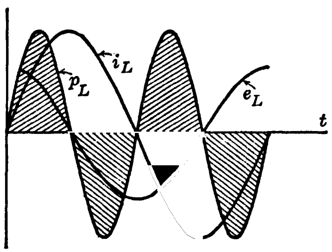


FIG. 15·17.

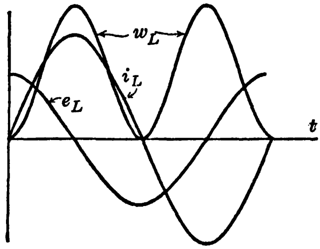


FIG. 15·18.

It will be observed that power is a *double-frequency* quantity with *average value zero.* Zero average power indicates no *permanent* absorption or delivery of energy.

As indicated by the power curve in Fig. 15·17 there is a *maximum* power. It is readily expressed by the power amplitude in equation 15·42.

$$(P_L)_{\max} = \tfrac{1}{2} X_L I_m^2 \qquad [15\cdot43]$$

This is the maximum time-rate of energy flow to (and from) the reactor. It is recognized as an index of the power performance of the reactor and is called simply the ***reactive power*** $P_L = (P_L)_{\max}$ although it is in truth also the *maximum* reactive power. No confusion ensues, because the *average* reactive power is *zero* and no values other than maximum and average are commonly encountered. The maximum *amount* of energy stored in the reactor is indicated by the *area* under the p-curve of Fig. 15·17 or may be computed from the relation $W = \frac{1}{2}Li^2$ which was introduced in the study of inductance. Evidently the maximum energy stored is

$$(W_L)_{\max} = \tfrac{1}{2} L I_m^2 \qquad [15\cdot44]$$

which is independent of frequency. Do not misconstrue this to imply that frequency is of *no consequence,* however; this maximum energy is

stored and unstored twice during each cycle and the *traffic in energy* is directly proportional to frequency.

Instead of finding $(W_L)_{max}$ from the area under the *p-t* curve, we may use $W_L = \frac{1}{2}Li_L^2$ to obtain a curve of *W-t* directly as in Fig. 15·18. Observe that the stored energy is maximum in Fig. 15·18 when power input p_L has just ceased (Fig. 15·17) and power output $(-p_L)$ is about to begin; the two aspects check as they should.

15·14. Mutual Reactance. The concept of inductive reactance is not confined to self-inductance L; it is equally applicable to mutual inductance L_M and may be expressed by

$$X_M = 2\pi f L_M \qquad [15\cdot45]$$

and

$$X_M = \frac{E_m}{I_m} \qquad [15\cdot46]$$

where $L_M = N_1N_2\mathcal{P}_M$ while E_m and I_m are the respective *amplitudes* of the voltage induced in one circuit by the sinusoidal current in a second circuit related to the first by the mutual permeance $\mathcal{P}_M$.

It should be observed that X_M, like L_M, *can* be a *minus* quantity in distinction from X_L which, like L, can have meaning only as a plus quantity. Otherwise, the same general properties enumerated for X_L are also possessed by X_M.

15·15. Resistance in A-C Circuits. Ohm's law applies to a-c circuits as well as to d-c circuits. More care must be exercised, however, to insure that *pure resistance only* is between the *apparent* terminals of the resistance; inductive effects in particular are likely to be overlooked and must be segregated from the pure resistance in an analysis of the circuit.

From $e = Ri$ we note that, so long as R is constant, e is exactly proportional to i from instant to instant and, except for scales, their oscillograms must be identical. There are no restrictions on the shape of the *i-t* plot; the *e-t* plot will be an exact duplicate and may even be the same curve, as in Fig. 15·19, if the scales are suitably selected. It follows that Ohm's law holds for the maximum values $E_m = RI_m$ and that the voltage will be *in-phase* with the current for any wave shape.

15·16. Resistance with Sinusoidal Current and Voltage. When $i = I_m \sin \omega t$ through a resistance, the emf is

$$e = Ri = RI_m \sin \omega t$$

or

$$e = E_m \sin \omega t$$

where

$$E_m = RI_m \qquad [15\cdot47]$$

Figure 15·20 gives the oscillogram.

15·17. Power and Energy for Resistance *R*. The energy taken by resistance is converted *entirely into heat* and, for any wave shape (Fig. 15·19) the power at any instant is computed by $p_R = e_R i_R$, $p_R = Ri_R^2$,

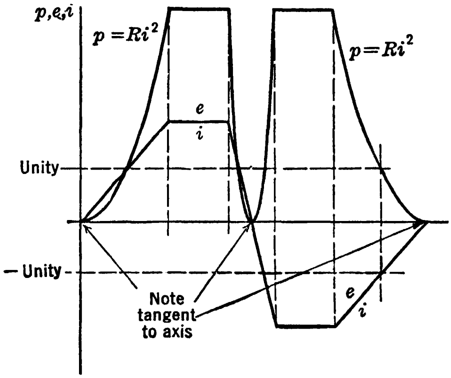


Fig. 15·19.

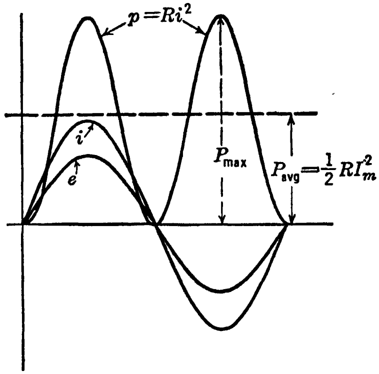


Fig. 15·20.

or $p_R = e_R^2/R$ as may be most convenient. For sinusoidal current (Fig. 15·20),

$$i_R = I_m \sin \omega t$$

$$p_R = Ri_R^2 = RI_m^2 \sin^2 \omega t$$

$$= \tfrac{1}{2}RI_m^2 - \tfrac{1}{2}RI_m^2 \cos 2\omega t \qquad [15 \cdot 48]$$

or

$$p_R = \tfrac{1}{2}E_m I_m - \tfrac{1}{2}E_m I_m \cos 2\omega t \qquad [15 \cdot 49]$$

The double-frequency term $\frac{1}{2}RI_m^2 \cos 2\omega t$ is reminiscent of the case for reactance power but here, because the constant term $\frac{1}{2}RI_m^2$ is equal to the amplitude of the sinusoidal term, the power curve cannot become negative; energy does not return from the resistor, and there is neither storage nor source of electrical energy in the resistor.

The average power of course is not zero. While it is possible to compute $P_{avg} = \dfrac{1}{T}\displaystyle\int_0^T p\,dt$, it is in this instance more shrewd to observe that the average of a sinusoidal quantity for one or more cycles is zero, which without further ado, leaves the constant component in equation 15·48 to be the average as shown in Fig. 15·20.

$$P_{avg} = \tfrac{1}{2}RI_m^2 \qquad [15 \cdot 50]$$

The *maximum* power is not particularly important but is readily found by observing (equation 15·48) that it comprises the sum of the

constant term P_{avg} (equation 15·50) and the greatest value or amplitude of the sinusoidal component, so that we write

$$(P_R)_{\max} = \tfrac{1}{2}RI_m^2 - \tfrac{1}{2}RI_m^2(-1) = RI_m^2 \qquad [15\cdot51]$$

Note that it would be unwise here to compute the maximum by the longer calculus routine of equating the derivative to zero.

15·18. Resistance and Inductive Reactance in Series. The practical considerations which justify the study especially of the *series* connection of R and L have already been expounded; they apply to alternating current as well as direct current, and we proceed here to the analysis for alternating current.

Again we turn to the now familiar basic equation $e = Ri + L(di/dt)$. Given the oscillogram of i-t to find e-t, the graphical procedure is merely an extension of that pursued for inductance alone and merits no special attention here. Given the oscillogram of e-t, to find i-t is *quite another matter*, and for general solution greatly exceeds the scope of this study. We propose therefore to confine our immediate study to the sinusoidal case where $i = I_m \sin \omega t$ and where the inductance, therefore, can be represented as inductive *reactance*.

15·19. For Sinusoidal Currents. Given:

$$i = I_m \sin \omega t$$

and

$$e = e_R + e_L = Ri + L\frac{di}{dt}$$

We find

$$e = RI_m \sin \omega t + \omega L I_m \cos \omega t$$

or

$$e = RI_m \sin \omega t + X_L I_m \cos \omega t \qquad [15\cdot52]$$

While the two components, $e_R = RI_m \sin \omega t$ and $e_L = X_L I_m \cos \omega t$ will give $e = f(t)$ by graphical process, as in Fig. 15·21, it is more informative to combine them by trigonometric process as was done in Chapter IX, p. 143.

From the *summation of sinusoidal quantities* developed there it follows that

$$e = E_m \sin (\omega t + \phi) \qquad [15\cdot53]$$

where

$$E_m = \sqrt{(RI_m)^2 + (X_L I_m)^2} = I_m\sqrt{(R^2 + X_L^2)} \qquad [15\cdot54]$$

and

$$\phi = \tan^{-1}\left(\frac{X_L I_m}{RI_m}\right) = \tan^{-1}\left(\frac{X_L}{R}\right) \qquad [15\cdot55]$$

From these three equations we establish three important truths about the series RL circuit, as follows.

1. The total voltage is sinusoidal.
2. The amplitude of this voltage is the square root of the sum of the squares of the resistance and reactance voltage amplitudes.
3. This voltage leads the current by the phase angle $\phi = \tan^{-1}(X_L/R)$.

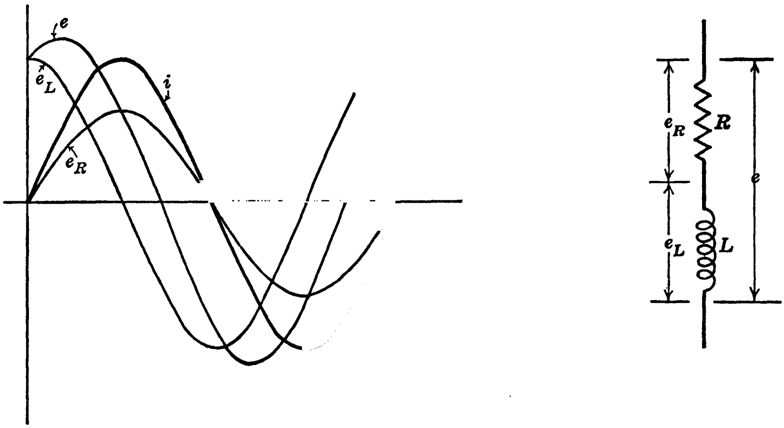


FIG. 15·21.

15·20. Impedance. In dealing with inductance carrying sinusoidal current, we obtained the concept of *reactance* as the ratio of maximum sinusoidal voltage across the inductance to maximum sinusoidal current through the inductance. We have also found that resistance, for *any* kind of current, can be expressed as the ratio of the maximum voltage across the resistance to the maximum current through the resistance.

Now that we have a combination of resistance and reactance we observe that equation 15·54, restricted to sinusoidal current and voltage by the nature of reactance, may be written

$$\frac{E_m}{I_m} = \sqrt{R^2 + X_L^2} \qquad [15\cdot56]$$

We may well inquire whether this ratio of the maximum sinusoidal emf across the RL combination to the maximum sinusoidal current through it may constitute another of these quantities having the ohmic dimension akin to resistance, and if so, we may further inquire whether there is any limit to the number of such quantities conceived in electric circuit analysis.

The answer to both inquiries is yes. At this point we introduce the concept of an ohmic quantity so general or all inclusive that no more

are required. For sinusoidal circuits we define the ratio of the maximum voltage to the maximum current for *any combination* of exclusively passive circuit elements (not sources of emf) as the ***impedance*** of that combination and represent it by symbol Z as follows.

$$Z = \frac{E_m}{I_m} \qquad [15\cdot57]$$

Impedance Z then is a generic term which applies to all a-c circuit quantities measured in ohms, including resistance and reactance. While the definition given here is restricted to *sinusoidal* voltage and current, we shall find eventually that by using the ratio of *effective values* of voltage and current even this restriction is removed. It is commonly feasible to compute impedance from a circuit analysis of the component resistance and reactance elements, but this is not always possible in the present state of the art. For the simple series combination of R and L considered here, we see from equations 15·56 and 15·57 that the impedance is

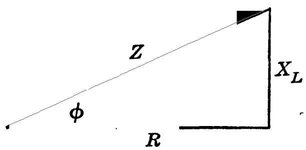


FIG. 15·22.

$$Z = \sqrt{R^2 + X_L^2} \qquad [15\cdot58]$$

The square root of the sum of the squares of two quantities is always suggestive of a right triangle which in this case is as shown in Fig. 15·22. This is known as an ***impedance triangle.*** The concept is encouraged by observing that angle ϕ also may be represented on the triangle in accord with equation 15·55, and we may write

$$\cos \phi = \frac{R}{Z} \qquad [15\cdot59]$$

$$\sin \phi = \frac{X}{Z} \qquad [15\cdot60]$$

The concept, of course, is restricted to the series circuit of resistance and reactance.

15·21. Power and Energy for Z with R and X_L. Again we observe that power $p = ei$ holds graphically and analytically, as always, for each instant of time. For sinusoidal current $i = I_m \sin \omega t$ we have found that

$$e = E_m \sin (\omega t + \phi) \qquad [15\cdot53]$$

These give for the instantaneous power product

$$p = E_m I_m \sin \omega t \sin (\omega t + \phi) \qquad [15\cdot61]$$

From trigonometry

$$\sin A \sin B = \tfrac{1}{2}\cos(A - B) - \tfrac{1}{2}\cos(A + B)$$

Let

$$A = (\omega t + \phi) \quad \text{and} \quad B = \omega t$$

Then

$$p = E_m I_m[\tfrac{1}{2}\cos\phi - \tfrac{1}{2}\cos(2\omega t + \phi)]$$

or

$$p = \tfrac{1}{2}E_m I_m \cos\phi - \tfrac{1}{2}E_m I_m \cos(2\omega t + \phi) \qquad [15\cdot62]$$

Here again we find in equation 15·62 the *double-frequency component* which we have previously encountered in each computation of instantaneous power for sinusoidal voltage and current. Note that it is always not only double frequency and sinusoidal, but that it *always has an amplitude equal to one-half the product of the associated voltage and current amplitudes.* Furthermore this amplitude is *independent of the phase angle* ϕ between e and i.

The *constant* component $\frac{1}{2}E_m I_m \cos\phi$ in equation 15·62, unlike that for resistance alone (equation 15·49), is in general *less* than the amplitude $\frac{1}{2}E_m I_m$ of the sinusoidal component. Clearly this constant component controls the *vertical position* of the sine-shaped $p = f(t)$ and in general brings the lower parts into the negative region as in Fig. 15·23.

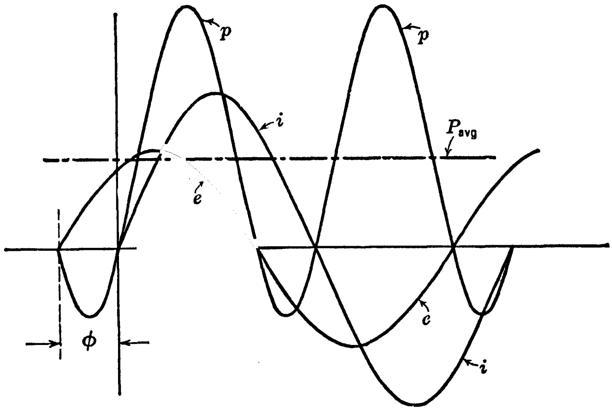


FIG. 15·23.

It is particularly useful to observe that the *size* and *shape* of the $p = f(t)$ curve are *independent* of the phase angle ϕ between e and i; that angle ϕ affects only the vertical and horizontal *position* of the curve.

The *average power* of course may be computed from $P_{avg} = \frac{1}{T}\int_0^T p\,dt$ but it is more readily obtained from the simple observation previously

exploited: ***the average value of any periodic quantity which varies symmetrically about a constant value is that constant value.***

The *constant value* about which $\frac{1}{2}E_mI_m \cos(2\omega t + \phi)$ of equation 15·62 *varies symmetrically* of course is $\frac{1}{2}E_mI_m \cos\phi$ and for this equation we write directly

$$P_{\text{avg}} = \tfrac{1}{2}E_mI_m \cos\phi \qquad [15\cdot63]$$

By substituting equations 15·57 and 15·59 we further find

$$P_{\text{avg}} = \tfrac{1}{2}(ZI_m)I_m\left(\frac{R}{Z}\right)$$

or

$$P_{\text{avg}} = \tfrac{1}{2}RI_m^2 \qquad [15\cdot64]$$

It is to be noted that this is the same as the average power for the resistance alone (equation 15·50). Remembering that the average power for reactance is zero we make a check observation that *conservation of energy* also indicates that the total average power supplied to the impedance must be the same as that supplied to the resistance.

15·22. A Practical Difficulty Created by Reactive Power. It is *not* to be inferred that the *zero* average value of reactive power eliminates it from any concern in the power supplied to an impedance.

Returning to the instantaneous power curve $p = f(t)$, we have observed that in general the curve has a below-axis or minus portion. The area under the $p = f(t)$ curve, of course, is $W = \int p\,dt$, which here represents the energy transferred to or from the impedance as in Fig. 15·24. The *plus* area W_1 represents energy delivered *to* the impedance, while the *minus* area W_2 represents energy delivered *by* the impedance. From previous study of the R and L components of the impedance which we are here considering, we know that only L can both take and return (store) energy. Consequently W_2 must come from the inductive reactor, and the energy which goes into the resistor for conversion to heat must be

$$W_h = W_1 - W_2$$

This "take and return" of energy by reactance introduces a very real and widely prevalent problem in the merchandising of a-c energy or "power" as it is commonly miscalled. Much of our present-day power apparatus inherently comprises a considerable component of inductive reactance. The burden of handling this reactive energy is little different for a *power company* than is the burden of handling return merchandise for the *merchant* who deals in material commodities instead of energy.

Suppose, for example, that we wanted 2 tons of coal repeatedly, but always ordered 3 tons and demanded that the third ton be taken back after delivery was completed. It is unlikely that this would be continued unless we agreed to finance the additional burden of transportation and labor supplied by the coal merchant. The situation is no different for the *power* merchant, and most power companies include in their rate schedule for industrial (and sometimes commercial) customers, bonus and penalty provisions to finance equitably the various amounts

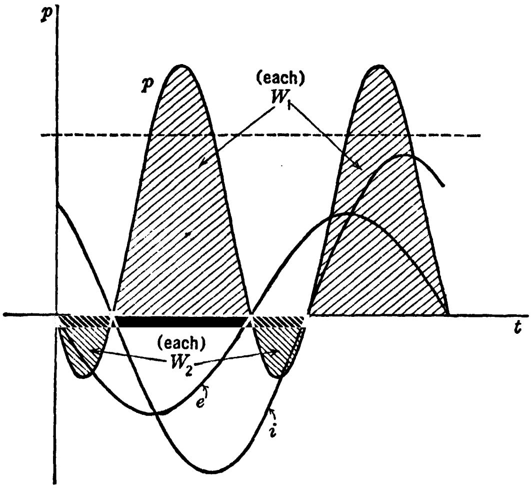


FIG. 15·24.

of this service required by different users. Further study of the subject will eventually be encountered by the student in connection with *power factor* and *power factor correction* but it is well here to comprehend something of the importance of reactance in the general picture of electric power merchandising beyond the fact that it takes no average power.

Let us observe now that W_2 in Fig. 15·24, the energy returned by the impedance to the source, is *not* the full energy stored in the reactance but only *part* of that energy. A typical example is illustrated in Fig. 15·25 where both the curve of instantaneous *total* power p (as in Fig. 15·24) and the curve of instantaneous *reactive* power p_L (as in Fig. 15·17) are shown. It is evident that the energy area $W_L > W_2$ and that only part of the returned reactive energy W_L *actually* goes out from the impedance. What then does become of the excess reactive energy $W_L - W_2$? The phase relations (timing) are always such as to permit it to be transferred to the resistance component of the impedance and to contribute to the i^2R heat energy. This is readily checked by constructing the

instantaneous heat power curve p_R of the resistance as in Fig. 15·25. Of course it is always true algebraically that $p = p_R + p_L$ and, during the transfer of energy $W_L - W_2$ from L to R, the supply of power p is *less* than the p_R taken by the resistance. It can be shown that the fraction of reactive energy which is transferred to R is

$$\frac{W_L - W_2}{W_L} = \phi \cot \phi \qquad [15\cdot65]$$

where ϕ is the phase angle in radians between e and i, i.e., the phase angle of the impedance. Since we have found that $R = Z \cos \phi$ and

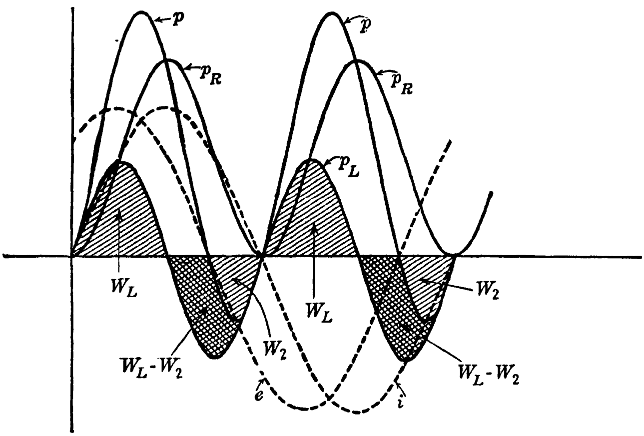


FIG. 15·25.

$X = Z \sin \phi$ (equations 15·59 and 15·60) we may substitute these for cotangent ϕ to give

$$\frac{W_L - W_2}{W_L} = \phi \frac{R}{X_L} \qquad [15\cdot66]$$

In terms of resistance power P_R and reactive power P_X:

$$\frac{W_L - W_2}{W_L} = \phi \frac{P_R}{P_X} \qquad [15\cdot67]$$

Between 0 and $\frac{\pi}{2}$, which are the practical limits for ϕ, the ratio runs from unity to zero. This relation is not of commercial import at present but cannot be overlooked if a clear understanding of power phenomena involving the prevalent case of an RL impedance is desired.

REFERENCES

1. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., Seventh Edition, pp. 5-232–5-243, 12-134–12-147.
2. KARAPETOFF, "The Electric Circuit," Chap. VI, Art. 21, p. 72, Art. 24.
3. ESHBACH, "Handbook for Engineers," John Wiley and Sons, p. 8-61.

QUESTIONS

15·1. Study the various aspects of a *parallel* connection of R and L as compared with the *series* connection, and prove that only the *series* connection correctly represents the relation of R and L as found in a simple coil.

15·2. Explain the significance of the *time constant* for a series circuit of R and L.

15·3. What per cent error would be involved by assuming that the final value of current in Fig. 15·3 is reached in time $t = 4T$?

15·4. Derive a relation for the initial rate of growth of current in a series RL circuit with suddenly impressed d-c voltage.

15·5. Construct and explain a relation for computing the initial rate of change of current in a series RL circuit which applies either to growth or to decay.

15·6. What factors at any instant determine the rate of change of current in the circuit of Fig. 15·5 for either switch position?

15·7. Prove that the initial rate of change of current in the circuit of Fig. 15·3 is the greatest rate of change.

15·8. If the time constant T of the circuit in Fig. 15·5 is kept the same by doubling the value of both R and L explain what changes will result in the behavior of the circuit, assuming the switch to be in each position for a period equal to 10 T?

15·9. The time constant T of the circuit of Fig. 15·5 may be doubled either by doubling L or halving R. Explain fully the difference in the result produced by these alternative means of doubling T for each switch position.

15·10. If the rate of decay of current in an RL circuit could actually be maintained at the initial rate, as conceived by the time constant, zero current of course could be reached in time T without exceeding the voltage applied during growth. Suggest a circuit for doing this, and explain the practical difficulties in producing actual apparatus even to approximate the desired performance.

15·11. A sinusoidal emf of given frequency is applied to a pure inductance. Explain the consequences of doubling the frequency of the emf, keeping maximum emf the same.

15·12. When a sinusoidal emf across an inductance has a positive maximum the current is zero, increasing. Is this true *only* for a *sinusoidal* emf? Explain fully.

15·13. Explain why it is insufficient to specify a reactor by *ohms* and *amperes* as in general may be done for a resistor.

15·14. Why is iron used in reactors—why not use a nonmagnetic core or *air* core to avoid any question about constant permeability?

15·15. Why are reactors often used instead of rheostats (adjustable resistors) for lighting control or theater dimmers?

15·16. Fig. 15·18 indicates that energy w_L has an *average* value *not zero*. How is this value related to *maximum* stored energy $(W_L)_{\max}$ and what is its significance?

15·17. If the average power taken by a reactor is zero why should a power company rightly object to the customer putting one on the line as he might for dimming

lights? (Consider carefully your reply to question 16 and the pertinent text in analyzing this situation.)

15·18. Explain why alternating voltage is not induced in the d-c control coil of the dimming reactor of Fig. 15.14.

15·19. Sketch a typical *B-H* curve and explain how the dimmer of Fig. 15·14 works.

15·20. Define *reactive power*, making clear why it is *not* an average value and why it is restricted to *sinusoidal* voltage and current.

15·21. Derive a relation which shows, for a series connection of R and L carrying sinusoidal current,

(*a*) The phase angle between e and i, $\phi = f(R,L,f)$.

(*b*) The maximum emf, $E_m = f(I_m,R,L,f)$.

15·22. What is impedance and why is it not *defined* as $\sqrt{R^2 + X_L^2}$?

15·23. Sketch the *p-t* curve for resistance carrying sinusoidal current and write the equation for it.

15·24. Derive an expression for the average power supplied to a series connection of R and L carrying sinusoidal current, and show that it is equal to that taken by the resistance alone.

15·25. Show why reactance is objectionable to the power merchant even though it takes no average power.

15·26. Why for a non-sine emf applied to a series connection of R and L is it difficult to compute the current, $i = f(t)$?

15·27. Show mathematically that, for a given amplitude of sinusoidal voltage and current, the instantaneous power $p = f(t)$ curve is sine-shaped and of identical size for all phase angles between the given voltage and current.

15·28. Derive equations 15·65 and 15·66.

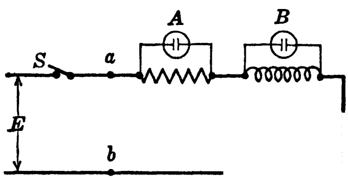
PROBLEMS

15·1. Given a series RL circuit with time constant $T = 0.001$ sec and a constant-voltage source, as in Fig. 15·3, compute the time required for:

(*a*) Current to grow from zero to within 0.1 per cent of its final or steady-state value.

(*b*) The voltage across L equal to that across R.

15·2. A d-c voltage of 110 volts is applied to the lecture demonstration circuit,



shown in the figure, consisting of 50 ohms resistance in series with inductance L. Neon lamps A and B each have:

Ignition voltage = 90 volts.
Extinction voltage = 70 volts.
Resistance approximately infinite.

(*a*) Compute the value of L which will make lamp A light 3 sec after voltage E is applied.

(*b*) Compute the time at which lamp B will extinguish after E is applied.

(*c*) Let a pure resistance of 950 ohms be connected across ab after i_L has reached ultimate value I. Compute the voltage across L at the instant switch S is opened.

15·3. A constant source of emf is suddenly applied to a series RL circuit having time constant $T = 0.02$, compute:

(*a*) The time required to supply equal amounts of energy to R and L.

(*b*) The amount of the energy in (*a*).

(*c*) The maximum power supplied to the inductance.

15·4. A typical design of the electromagnet of Problem 13·9 includes the following data:

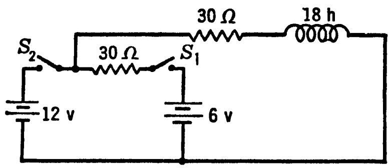
$$E = 220 \text{ volts} \qquad \phi = 4280 \text{ kilolines}$$
$$P = 40 \text{ watts} \qquad N = 16{,}500 \text{ turns}$$

(*a*) Compute the time constant of the magnet.

(*b*) What value of resistance must be shunted across the magnet during the opening of the supply switch (as in Fig. 15·8) to prevent any voltage in the magnet from exceeding three times the rated value?

(*c*) How much energy must this resistor be designed to dissipate?

15·5. Beginning with both switches open in the accompanying circuit, S_1 is closed and remains so. After a period equal to twice the time constant, S_2 is alternately snapped closed and open, remaining in each position for a period equal to the time constant of the circuit during the prevailing period.



(*a*) Compute the *time constant* and the objective or *steady-state* value of current for *each* position of S_2 when S_1 is closed.

(*b*) Sketch with care the graph $i_L = f(t)$ for the first two cycles of operation of switch S_2.

(*c*) Compute the current at each of the first four operations of S_2.

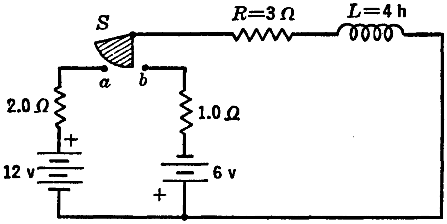
(*d*) Compute the rate of current change initiated by S_1 and by each operation of the switch as in (*c*).

(*e*) Compute the energy taken from the battery before S_2 is first closed and the per cent of this dissipated in heat.

(*f*) If S_2 is operated for an indefinitely long time will i_L *settle down* to varying between *fixed* values? Discuss.

15·6. Repeat Problem 5 with values of the two battery voltages interchanged and with *each* switch interval equal to the time constant for that interval.

15·7. In the accompanying circuit, switch S can be moved to switch from a to b without opening the RL circuit.



(*a*) Compute the time constants T_a and T_b for the respective switch positions.

(*b*) Sketch the circuit for each switch position and compute the ultimate value of current for each.

(*c*) What value of current will be reached in time $t = 2T_a$ after starting to operate the circuit with switch moved to a from the open position?

(*d*) If the switch is quickly moved from a to b when $i = 2$ amp through RL, how long will it take the current to reach zero?

15·8. In order to show the curves of Fig. 15·9 on an oscillograph the switch of Fig. 15·8 is operated repeatedly at a frequency of 60 cycles per second, which is conveniently obtained and high enough to avoid serious flicker of the oscilloscope picture.

(*a*) Derive an expression for the time interval $T_1 = t_3 - t_2 = f(R,L)$ and $T_2 = t_2 - t_1 = f(R,R',L)$ in Fig. 15·9 which will make

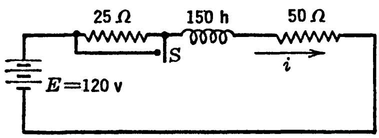
$$I_1 = 0.731I$$

$$I_2 = 0.269I$$

(*b*) It is desired to make $T_1/T_2 = 2$. Determine what elements in the construction of the circuit control this ratio and how they must be related to give the desired result.

(*c*) Given $L = 10$ henrys limited to 300 volts maximum, compute the values of R and R' required to produce $T_1/T_2 = 2$ for the 60-cycle switching rate.

(*d*) Compute the largest value of E (Fig. 15·8) that is allowable in (*c*).



15·9. The above circuit represents the elements of a type of current regulator which, by the magnetic opening and release of the spring-closed switch S, regulates current i within specified limits of variation, even though voltage E (given *constant* here) may be somewhat inconstant. If i is permitted to vary ± 1 per cent from the arithmetic mean of the possible extreme values, compute for the given circuit values:

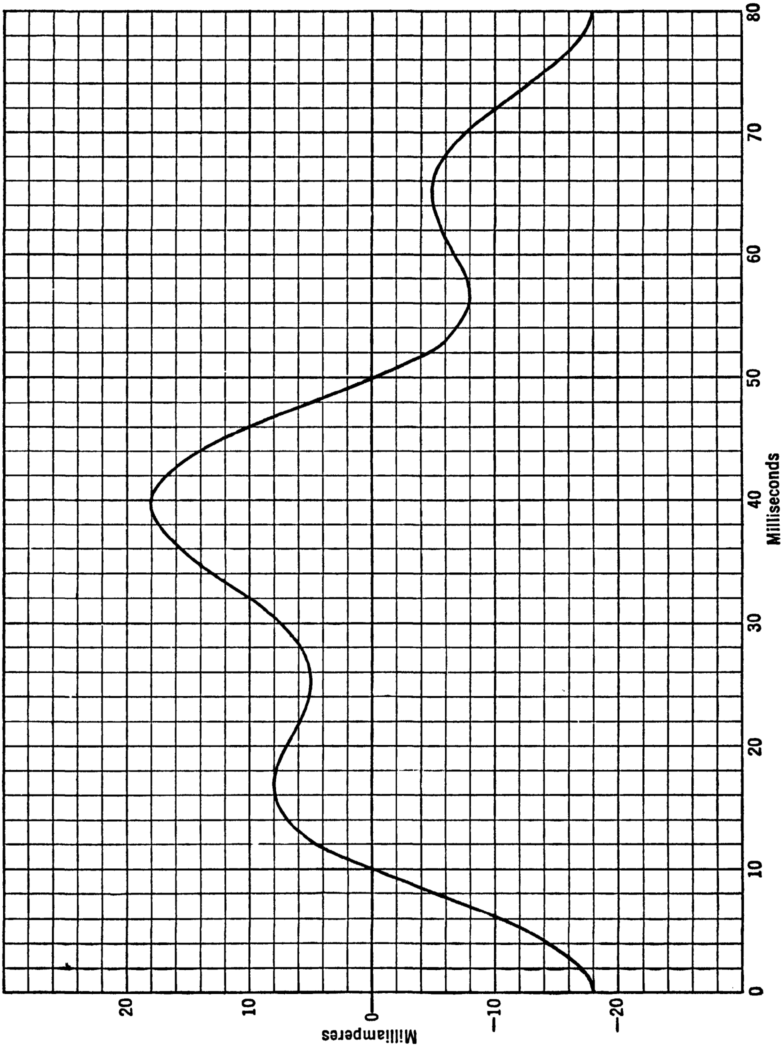
(*a*) Time T_1 during which the switch must be closed.

(*b*) Time T_2 during which the switch must be open.

(*c*) Frequency of switch operation in cycles per second.

15·10. (*a*) Given the attached oscillogram of an alternating current through a pure inductance of 55 henrys. Determine the emf across the inductance for at least twelve instants and plot the *e-t* curve. Observe that the points of maximum current and maximum *di/dt* are especially desirable points to deal with—but be sure you know why this is so before proceeding.

(*b*) Compute the frequency in cycles per second.



15·11. An iron-core inductance of 15 henrys is desired for use on a 1000 cycle per second sinusoidal emf of maximum value 472 volts. A sheet steel toroidal core is available with section 4 sq cm and length 200 cm. The *B-H* curve is given in Chapter

XI on Magnetics and shows that for approximately constant μ, $B = 9$ kilogauss is about the maximum flux density to be permitted. Compute:

(a) Maximum current which the inductance must carry ($E_m = I_m X$).

(b) Number of turns required ($L = N^2 \mathcal{P}$, MKS units).

(c) Maximum density of flux created by the current of (a) and turns of (b).

This inductance is used in the plate circuit of a vacuum-tube oscillator so that it must carry direct current as well as alternating current. Compute:

(d) Amperes of direct current I_{dc} which may be superimposed on the alternating current without exceeding the permissible flux density.

(e) If space limits the turns to 6000, what is the longest air gap that can be inserted in the core and still have specified 15 henrys inductance?

(f) Repeat computation (d) for the construction of (e) and compare, i.e., find ratio

$$\frac{I_{dc} \text{ with gap}}{I_{dc} \text{ no gap}}$$

15·12. Determine:

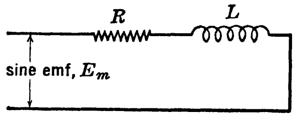
(a) Inductive reactance X_L.

(b) Impedance Z.

(c) Maximum value of the current I_m.

(d) Phase angle ϕ between the applied voltage and the current—state whether lead or lag (i.e., current with respect to voltage).

(e) Equation of applied voltage, using all known and determined constants. Let $t = 0$ when $i = I_m$.



$E_m = 424$ volts
$R = 12$ ohms
$L = 0.0424$ henry
$f = 60$ cycles/sec

15·13. An air-core coil has a resistance of 300 ohms and an inductance of 1 henry. A sine wave generator with an output voltage of 500 volts maximum is connected to the coil, and it is then found that the current measures 1 amp maximum.

(a) What is the frequency of the generator?

(b) What is the average power taken by the coil?

(c) What is the maximum energy stored in the coil inductance?

15·14. A series circuit of $R = 100$ ohms and $L = 60$ millihenrys takes 8 watts average power when connected to a voltage

$$e = 120 \sin \omega t$$

(a) What is the frequency of the voltage?

(b) What is the equation for the current?

(c) What is the reactive power?

(d) What is the maximum energy stored in L?

(e) What is the maximum instantaneous value of the heat power produced by R?

15·15. Given a series connection of resistance and inductance with current and applied voltage

$$i = 12 \sin 377t$$

$$e = 240 \sin \left(377t + \frac{\pi}{6}\right)$$

Compute:

(a) Impedance.

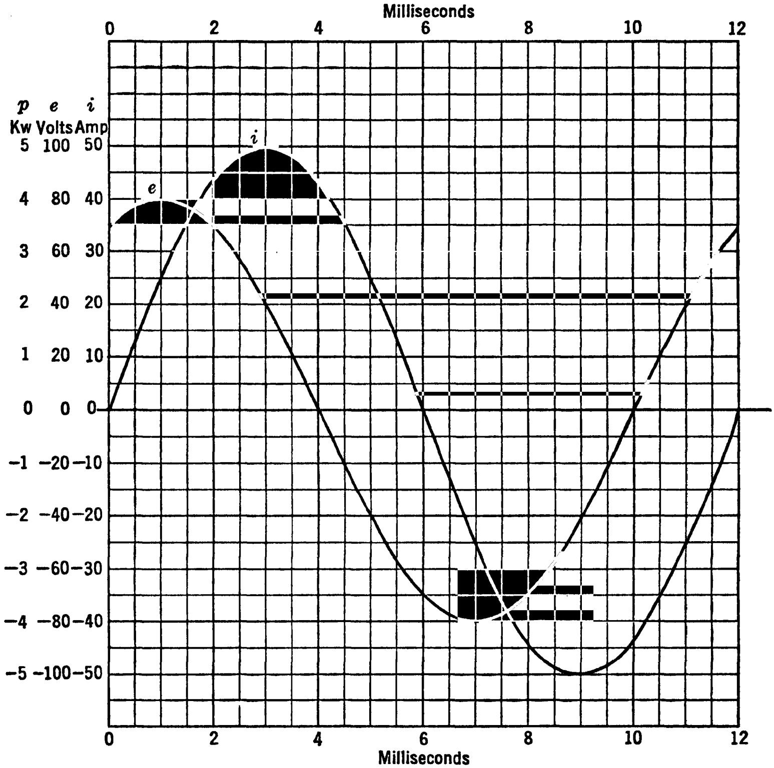
(b) Angle of phase displacement between e and i.

(c) Resistance.
(d) Reactance.
(e) Inductance.
(f) Maximum instantaneous power supplied to the impedance.
(g) Maximum instantaneous power *returned* by the impedance.
(h) Maximum energy stored in the inductance.
(i) Average power (in watts) converted to heat.

Sketch:

(j) Curves of e, i, and p versus t.

15·16. Given the accompanying sine waves of voltage and current for a series connection of R and L.



(a) By graphical process analyze e into the component emf's e_R and e_L across the resistance and inductance respectively.

(b) By analytical process find $e_R = f(t)$ and $e_L = f(t)$.

(c) Compute R, X_L, L, and Z; check $\phi = \tan^{-1}(X_L/R)$ with the graph.

(d) Obtain graphically the curves of total instantaneous power p and the components p_R and p_L.

(e) Find the equation for $p = f(t)$ and compute the average power P_{avg}.

(f) Compute the energy in joules converted into heat during one cycle.

(*g*) Compute the total (not the net) energy *supplied* to impedance Z from the power supply during one cycle, c.f. W_1 of Fig. 15·24.

(*h*) Compute the total of energy, c.f. W_2 of Fig. 15·24, actually *returned* from impedance Z to the power supply during one cycle and check against the data from (*f*) and (*g*).

(*i*) Compute the ratio of energy in (*f*) to energy in (*g*), that is, the fraction of the gross energy *supplied* which is useful (as heat).

(*j*) Compute the ratio of energy in (*f*) to the total energy *traffic* ($g + h$), that is, the fraction of the total energy traffic which is useful (as heat).

(*k*) Compute the energy transferred from L to R during one cycle.

(*l*) Compute from (*k*) the fraction of energy stored in L which is transferred to R. Check this value against $\phi \cot \phi$ mentioned in the text.

CHAPTER XVI

ELECTROSTATICS

16·1. Historical Note. Electrostatic phenomena as manifested by the attraction of bits of paper and the like by rubbed amber, were observed at least as early as 300 B.C. Nothing of particular note transpired until VonGoericke constructed a frictional electric machine in 1672. This was followed by Sir Isaac Newton's practical glass-plate machine in 1676 and by the Leyden jar in 1745. These preceded Franklin's famous kite experiment of 1752 and Coulomb's pronouncement in 1785 of the inverse square law for electrical charges.

The Leyden jar was thus born in the dark age of electricity when means for quantitative observation were of the crudest kind. We find some entertaining accounts of the earliest experiments in electrostatics. An electric charge was commonly measured by the jolt received from its discharge through the long-suffering experimenter's body. By successive application of an electrophorous or other "generator" the Leyden jar could be charged to what was then a new high in "joltage." That the Leyden jar and its progeny of "condensers" should have been conceived to concentrate or condense electric charge was at the time a reasonable interpretation of the observers' experience.

In the light of our present knowledge, incomplete as it is, we have reasonable evidence that the condenser concept is erroneous, that *no electric charge is stored in the Leyden jar.* In consequence, the term condenser is gradually giving place to ***capacitor*** to denote a structure designed especially to embody the property known as *capacitance. Permittor* and *permittance* will also be found occasionally in the literature.

16·2. Charging a Capacitor. When the Leyden jar capacitor was used with so-called *static* electricity, the electric circuit, being of indefinite and intangible dimension through air and ground paths, was not readily observed. Having since learned that *static* and *galvanic* electricity (as well as *magnetism*) are but different aspects of electronic phenomena, we commonly use capacitors in galvanic or wired circuits today. In such use observation and measurement of the phenomena are readily made, and some earlier misconceptions become apparent.

Let the capacitor C be connected to a source of emf E as in Fig. 16·1. Let meters be placed at the terminals QQ to measure the electrons or

coulombs of electricity which pass through them following closure of switch S. It is an experimental fact that current flows for a brief interval and then ceases. A difference of potential equal to that of the battery appears across the capacitor terminals and remains there following cessation of current flow. The Q-meters (ballistic galvanometers possibly) indicate that the precise number of coulombs of electrons makes *exit* at one terminal as *enters* at the other terminal. It follows that ***no change occurs in the amount of the electron content of the capacitor.*** This is an important observation because it is not unusual, even today, to find it stated that charging the capacitor means putting a charge of electrons

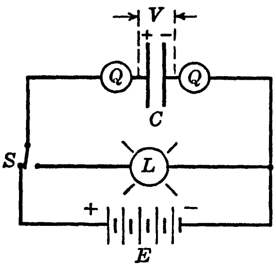


FIG. 16·1. Charging a capacitor.

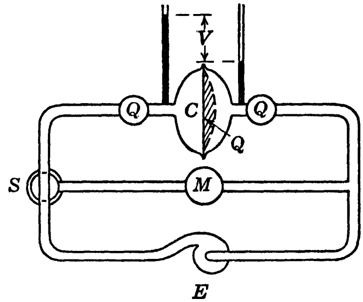


FIG. 16·2. Hydraulic analog for a capacitor during charge.

or electricity into the capacitor. It cannot be too strongly emphasized that this is *an erroneous concept,* as shown by the simple but adequate experiment just cited.

The *charging* of a capacitor, while adding nothing to the electrical charge or electronic content of the capacitor, does deliver to the capacitor a charge of *energy.* This is readily demonstrated by moving switch S to disconnect the battery and substitute lamp L which, if of suitable rating, will light during the brief time required to use up the electrical energy stored in the capacitor.

There is nothing unique, mysterious, or even peculiar about such phenomena; we wind a watch to store energy in it and well know that we do not add substance to the watch in thus "charging" it. The same is true when we set a spring trap, cock an air rifle, and perform many acts in everyday life which involve *storing energy by elastic deformation* of springs, rubber bands, and the like.

Of course this is not the only mechanism for storing energy; we store energy in pumping up a tire by increasing the amount of air in it; we store energy in the standpipe of a water supply system by increasing the amount of water in it. The point to be made clear, however, is that

the charging of a capacitor is *not* comparable to these other phenomena which involve *increase of substance* in the storing of energy.

16·3. An Analog. It may be found helpful to observe how the capacitor concept can be exemplified in hydraulic apparatus. Figure 16·2 shows the analogous hydraulic elements of the electric circuit of Fig. 16·1. Capacitor C is represented by an elastic (rubber) diaphragm clamped between two cuplike containers with pipe terminals leading through water meters QQ. Centrifugal pump E provides the source of pressure or potential difference and stopcock S the "switch." The system is completely filled with water to simulate the always electron-filled electric circuit.

When pump E is running, valve S is opened, and water is displaced all along the pipes. The diaphragm is displaced into the dotted position until a difference of pressure V, equal to E of the pump, is attained. Each water meter Q records the same number of gallons, and no water has been added to device C. Energy is stored in the diaphragm of C and can be utilized to run a water motor for a short time, if desired.

16·4. Quantitative Relations. What may be called the *law of capacitance* is well known as represented by the equation

$$Q = CE \tag{16·1}$$

We have observed experimentally that the *traffic count* Q of electrons into the one terminal of a capacitor (and *out of the other terminal*) is accompanied by a change E in the potential difference between the terminals of the capacitor. As shown by equation 16·1, the change in voltage is found by experiment to be directly proportional to this electron displacement or traffic count.

The constant of proportionality $C = Q/E$ characterizes the particular capacitor and functionally defines the property ***capacitance*** as *the number of coulombs of electricity moved through * the capacitor per volt change of potential difference between terminals.*

The MKS unit of capacitance is the ***farad*** which is measured in coulombs per volt. Practical capacitors are commonly of such size that measure in microfarads (10^{-6} farad, abbreviated μf) or even micromicrofarads (10^{-12} farad, abbreviated $\mu\mu$f) is most convenient.

* Meaning *into and out of.* No word is entirely adequate to express clearly the displacement which occurs all along a path through something without implying the *complete passing through* of an individual member of the movement. In this case no individual electron completes the journey from one terminal to the other, even approximately. Care also must be exercised to avoid implying the number of electrons which *comprise the substance* of the capacitor as might be suggested by "throughout."

16·5. Capacitor Construction. A ***capacitor*** consists of any combination of dielectrics and conductors assembled for the purpose of storing electric energy. In the interest of clarity we shall consider primarily capacitors with *solid* and *liquid* or tangible dielectrics which constitute the majority of commercial capacitors. In general these dielectrics comprise our best insulators and will be discussed later. Gaseous and space dielectrics will be considered as special cases of the general concept here developed.

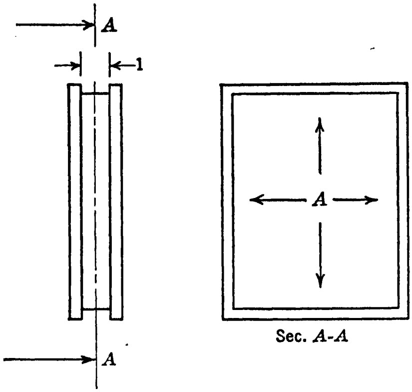


FIG. 16·3. Construction of sandwich type capacitor.

Let us first consider the familiar *sandwich* construction shown in its simplest form in Fig. 16·3. The dielectric is placed between two metallic terminal plates as the ham is placed between two bread slices for a ham sandwich. In much the same way as the ham is the major element which identifies and characterizes the sandwich, the dielectric of a capacitor is the major element which identifies and characterizes the capacitor. So long as the terminal plates are good conductors in good contact with the dielectric, their exact material or thickness is unimportant. This too may be said of the terminal bread slices of the ham sandwich which function primarily to provide satisfactory contact with the ham.

This point of view is stressed because the material and dimensions which determine the capacitance of the capacitor are primarily those of the dielectric and not of the plates.

16·6. The Constructional Relation. Capacitance depends upon the *material* of the dielectric, its *thickness* and *sectional area.* Like the magnetic permeance relation $\mathcal{P} = \mu_s\mu_0 A/l$, we find for capacitance

$$C = \kappa_s\kappa_0 \frac{A}{l} \qquad [16\cdot2]$$

where A is the sectional area and l the length of the dielectric *path* between contact plates. Constants κ_s and κ_0 are respectively the *specific* dielectric constant and the *absolute* dielectric constant for free space.

While it might be expected that these would be termed the "capacitability," "capacitivity," or "capacivity" of the dielectric such terms are not in use. The term *permittivity* is used somewhat, but either *dielectric*

constant or the time-honored *specific inductive capacity* is more common. In the same manner as for permeability, $\kappa_s\kappa_0$ represents the capacitance of a unit cube of the given material as may be shown by writing equation 16·2 for $A = 1$ and $l = 1$, which gives: $C = \kappa_s\kappa_0$.

In MKS units $\kappa_0 = 8.84 \times 10^{-12}$ farad per meter. The specific constant κ_s of course is without dimension, and is unity for free space. It is practically unity for air. Values of κ_s for many materials are available from various handbooks, International Critical Tables, and the like. Unlike ferrous permeability, the dielectric constant is reasonably independent of the potential gradient. Like resistivity, however, there are materials and conditions for which considerable variation is experienced. Some typical values of κ_s are as follows.

Material	κ_s	Material	κ_s
Bakelite (sheet)	4.5 to 5.5	Paper (dry Kraft)	3.5
Castor oil	4.7	Paraffin	2.1 to 2.5
Chlorinated diphenyl	4.95	Polystyrene	2.6
Fiber	2.5 to 5	Porcelain (wet proc.)	7
Glass	5.5 to 9.0	Pressboard (oiled)	5
Isolantite	6.1	Pyrex	4.8
Mica (India)	7.0 to 7.9	Steatite	5.9
Oil (mineral)	2.23	Water	81
		Wood	2.5 to 7.7

16·7. Capacitor Structures. Capacitors are by no means always of the simple sandwich type. The assembly shown in Fig. 16·4 is a com-

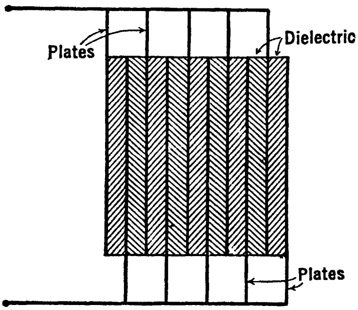


Fig. 16·4. A multiple element sandwich type capacitor.

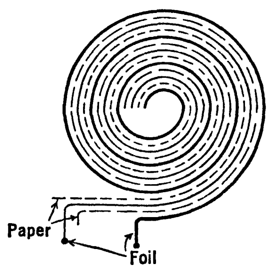


Fig. 16·5. A rolled type capacitor.

mon one which, while seemingly in series, is actually a parallel connection with the notable virtue of an economy of contact plates.

The *rolled capacitor* shown in Fig. 16·5 is very popular because it is easy to assemble by machine. These are especially common in radio

receivers. When noninductive capacitors are required, the rolled type of course is not suitable. An entirely different class of capacitors known as *electrolytic capacitors* is usually made in this form. The dielectric is a very thin film formed by electrochemical action on the surface of the anode, which is commonly made of aluminum or tantalum foil. The extraordinarily large capacitance which can be obtained in a small volume outweighs the disadvantages for many applications. Units of 8 and 16 μf, good for 600 volts test, and no larger than the thumb are now standard components of communication and electronic equipment.

For high-voltage work *oil-immersed capacitors* are usual. The use of chlorinated diphenyl has found considerable favor because of the noninflammable character of the liquid as well as other desirable properties.

Especially in the fields of communication and electronics adjustable or *variable capacitors* are necessary. Most of these have air for the principal dielectric. Because of the mechanical limitations of any such structure it is impossible wholly to eliminate solid dielectrics, and their effect on the capacitor characteristics is seldom negligible at radio frequencies. The design of these capacitors is quite diverse. Some of the most common are represented in Fig. 16·6. Style (*a*) is a compression type made like a book with leaves of spring brass alternating with mica.

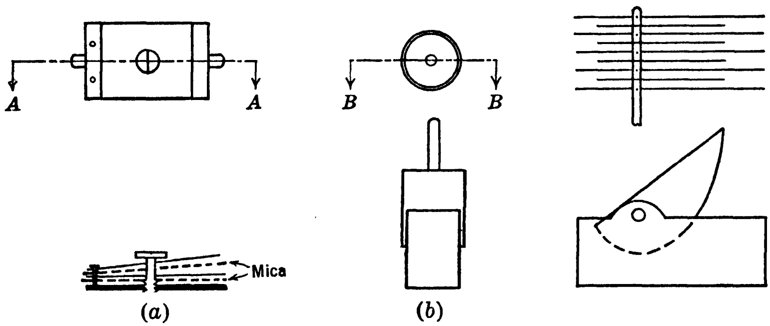


FIG. 16·6. Typical variable capacitors.

As the compression screw closes the book, the air dielectric in series with the mica is reduced and the capacitance increased. These are commonly made with an area less than 1 sq in. and have capacitances of only a few micromicrofarads. Style (*b*) comprises two concentric cylinders which may be telescoped so as to increase the cross section of air dielectric in the constant radial gap between cylinders. Adjustment is made by moving a plunger attached to one cylinder and supported by a compressible sleeve with a lock nut. Style (*c*) is the familiar rotary *variable condenser* which comprises the station selector or tuner of most

radio receivers. It is much like the capacitor of Fig. 16·4 except that one set of alternate plates is movable as a unit in such manner as to maintain constant air gap or plate spacing while varying the effective cross section of dielectric between plates of opposite polarity.

16·8. Capacitance of Nonuniform Sectional Area. The capacitance of a single-conductor cable between conductor and metallic sheath is an example of many cases where capacitance must be computed by integration. A cross section is shown in Fig. 16·7.

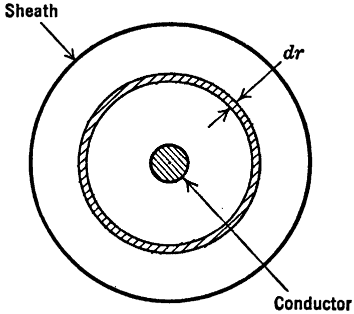


Fig. 16·7.

For this the capacitance elements are in series radially from conductor to sheath. They may be conceived to comprise cylindrical concentric layers of dielectric of thickness dr and section $A = 2\pi rl$ comparable to the yearly layers of growth in a tree which appear in the well-known concentric rings of the tree section. Capacitance elements in series of course do *not* integrate directly. Probably the best attack is by the application of Kirchhoff's ΣE law, recognizing that Q is the same for all "rings."

$$E = \int de = \int \frac{Q}{\kappa A}\,dr = \frac{Q}{2\pi l\kappa}\int \frac{dr}{r}$$

or

$$C = \frac{Q}{E} = \frac{2\pi l\kappa}{\displaystyle\int \frac{dr}{r}} \qquad [16\cdot3]$$

Where l is length of cable, κ is absolute dielectric constant of insulation, and r is radius from center of conductor to any point between conductor and sheath.

16·9. Energy in a Capacitor. The energy stored in a capacitor is computed from the usual basic relation $W = \int p\,dt$ or $W = \int ei\,dt = \int e\,dq$. For the special case of the capacitor, $q = Ce$ wherein C is a constant, so that $dq = Cde$, and we find

$$W_C = \int_0^{E_C} e(Cde) = \tfrac{1}{2}CE_C^2 \qquad [16\cdot4]$$

Substituting $Q = CE$, two other forms are found which are useful on occasion.

$$W_C = \tfrac{1}{2}QE_C \qquad [16\cdot5]$$

$$W_C = \frac{1}{2}\frac{Q^2}{C} \qquad [16\cdot6]$$

The MKS units, of course, are W_C joules, Q coulombs, C farads, and E_C volts.

It is helpful to visualize the factor ½ in the above by means of the simple graphical concept of Fig. 16·8 which is a plot of $q = Ce$. The integral $W = \int e\,dq$ of course is the area under the curve. This is clearly a triangle with sides E and Q so that the area is one-half that of the rectangle having sides E and Q. Thus $W = \frac{1}{2}QE$ as derived above.

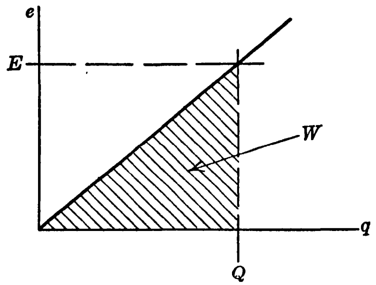


FIG. 16·8.

16·10. Electric Circuits with Capacitance. If it is realized that capacitors in electric circuits do not violate the basic principles expressed in Kirchhoff's laws, there is no occasion for uneasiness when capacitors have to be dealt with in ordinary electric circuits.

While the capacitance relation may be written $i = C(de/dt)$ as well as $q = Ce$, for constant capacitance, it is to be observed that unidirectional currents (d-c) cannot flow through capacitors continuously and that in these cases it may suffice for some purposes merely to determine the final status of an electric circuit in consequence of some change, say, in the connection of the circuit. We would then use $q = Ce$ rather than $i = C(de/dt)$, and modify Kirchhoff's junction relation to read $\Sigma q = 0$ rather than $\Sigma i = 0$. This modification is entirely permissible because it is merely an integrated form ($q = \int i\,dt$) of the basic fact that electrons cannot accumulate or be created or destroyed at a junction in an ordinary electric circuit.

Let us consider how we might go about the application of these relations to some specific problems.

16·11. Capacitance Alone in D-C Circuits. Let two capacitors C_1 and C_2 be connected in series as shown initially in Fig. 16·9a. Capacitance C_2 is uncharged and C_1 is charged to potential E_1 with the polarity shown. When switch S is closed, current will flow to the integrated

amount ΔQ before equilibrium is established. Assume the conventional (not electron) direction to be as in Fig. 16·9*b* and let E_1' and E_2' represent the final emf's with polarities assumed as shown in Fig. 16·9*c*.

From $\Sigma E = 0$

$$-E_1' - E_2' = 0 \qquad [16\cdot7]$$

From $q = Ce$

$$\Delta Q = C_1 \Delta E_1 = C_1(E_1 - E_1') \qquad [16\cdot8]$$

$$\Delta Q = C_2 \Delta E_2 = C_2(0 - E_2') \qquad [16\cdot9]$$

It is imperative in equations 16·8 and 16·9 to recognize that, for the assumed direction of ΔQ, the accompanying ΔE's *must be plus.* If this

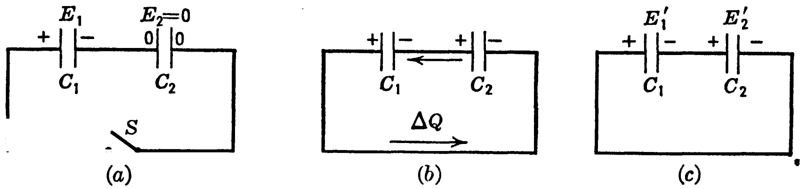


Fig. 16·9.

were not done capacitances C_1 and C_2 would be forced to absorb the minus sign and thereby become physically meaningless. It is not difficult to insure that the ΔE's be plus.

First we note that the direction assumed for ΔQ in Fig. 16·9 is, for C_1, from − to +, making C_1 function as an *energy source.* This discharges the capacitor and reduces its voltage from the initial E_1 of Fig. 16·9*a* to the final E_1' of Fig. 16·9*c*. To make ΔE the required *plus* value it is then clearly necessary to subtract the smaller E_1' from the larger E_1. Thus we reason for equation 16·8 that $\Delta E_1 = E_1 - E_1'$ and *not* $E_1' - E_1$. The same reasoning applies to capacitor C_2 and equation 16·9 although here, since we cannot actually discharge the already discharged capacitor, it is clear that E_2' must come out with a *minus* value unless ΔQ should prove to be actually with direction opposite to that assumed. Of course there is no objection to assuming for E_2' the opposite polarity to that shown in Fig. 16·9*c*, provided we also recognize that ΔE_2 becomes $E_2' - 0$.

It is possible to solve the problem by assuming also the reverse of the polarity shown for E_1' in Fig. 16·9*c*, but this assumption implies discharge of C_1 from E_1 to zero followed by charge from zero to E_1'. This needless complication is avoided if we always *assume* for each capacitor *no change in polarity* during displacement ΔQ and depend on the algebraic sign to divulge the true polarity.

Solving the equations 16·7 to 16·9 for the quantities shown in Figs. 16·9*b* and *c*, we find

From equations 16·8 and 16·9

$$\Delta Q = C_1E_1 - C_1E_1' = -C_2E_2' \qquad [16\cdot 10]$$

Substituting equation 16·7

$$C_1E_1 = C_1E_1' - C_2(-E_1')$$

and factoring

$$C_1E_1 = (C_1 + C_2)E_1' \qquad [16\cdot 11]$$

Solving

$$E_1' = E_1 \frac{C_1}{C_1 + C_2} \qquad [16\cdot 12]$$

Substituting equation 16·7

$$E_2' = -E_1' = -E_1 \frac{C_1}{C_1 + C_2} \qquad [16\cdot 13]$$

From equation 16·10

$$\Delta Q = -C_2E_2' = E_1 \frac{C_1C_2}{C_1 + C_2} \qquad [16\cdot 14]$$

Note carefully that the two-capacitor case is a very special one because ***any circuit restricted to two elements may at will be conceived to be either a series or parallel connection of the two elements.*** The method used here differs from that frequently presented in that it is readily applied to more than two capacitors, even to a network, if need be, and does not depend upon the special properties of two-element circuits.

To be assured that our attack is not confined to two capacitors we should at least consider the series and the parallel connection of three capacitors. The intermediate and final circuit conditions like those given by (*b*) and (*c*) in Fig. 16·9 will be indicated here in one figure (*b*), it being understood that ΔQ then shows the displacement that *has* taken place to produce the indicated voltages.

Series Circuit. Let us consider first the series circuit of Fig. 16·10.

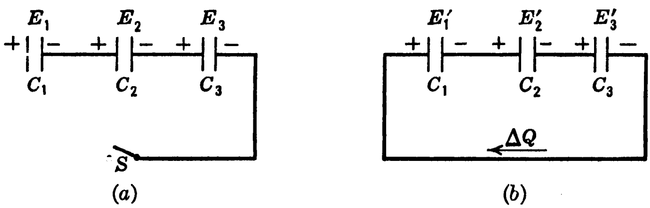


FIG. 16·10.

From $\Sigma E = 0$
$$E_3' + E_2' + E_1' = 0 \qquad [16\cdot15]$$
From $q = Ce$
$$\Delta Q = C_1 \Delta E_1 = C_1(E_1' - E_1) \qquad [16\cdot16]$$
$$\Delta Q = C_2 \Delta E_2 = C_2(E_2' - E_2) \qquad [16\cdot17]$$
$$\Delta Q = C_3 \Delta E_3 = C_3(E_3' - E_3) \qquad [16\cdot18]$$

Rewriting each of the above ΔQ equations, we obtain

$$\frac{\Delta Q}{C_1} = E_1' - E_1$$
$$\frac{\Delta Q}{C_2} = E_2' - E_2$$
$$\frac{\Delta Q}{C_3} = E_3' - E_3$$

Summing the above gives

$$\frac{\Delta Q}{C_1} + \frac{\Delta Q}{C_2} + \frac{\Delta Q}{C_3} = (E_1' + E_2' + E_3') - (E_1 + E_2 + E_3)$$

By substituting equation 16·15 and solving for ΔQ

$$\Delta Q = \frac{-(E_1 + E_2 + E_3)}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3}} = -\frac{\Sigma E}{\Sigma \dfrac{1}{C}} \qquad [16\cdot19]$$

The minus sign of course indicates that the *actual* direction of ΔQ is opposite to that *assumed* in Fig. 16·10*b*.

Thus we observe that the displacement is the same as would be produced by application of the net voltage (that across the open switch) to a single capacitance equivalent to the series combination. This of course is due to the simple linear relations which apply to the phenomenon and might be obtained by application of the superposition principle.

Solutions for E_1', E_2', and E_3' are now readily found and are left to the student.

Parallel Circuit. Considering now the parallel circuit of Fig. 16·11:

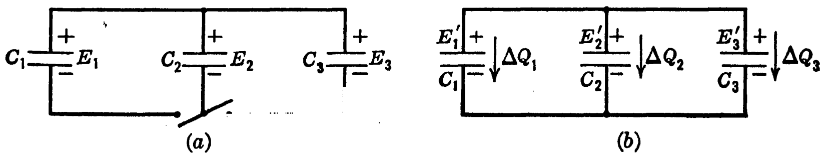


FIG. 16·11.

From $\Sigma Q = 0$

$$\Delta Q_1 + \Delta Q_2 + \Delta Q_3 = 0 \qquad [16\cdot 20]$$

From $\Sigma E = 0$

$$E_1' = E_2' = E_3' = E' \qquad [16\cdot 21]$$

From $q = Ce$

$$\Delta Q_1 = C_1 \Delta E_1 = C_1(E' - E_1) \qquad [16\cdot 22]$$

$$\Delta Q_2 = C_2 \Delta E_2 = C_2(E' - E_2) \qquad [16\cdot 23]$$

$$\Delta Q_3 = C_3 \Delta E_3 = C_3(E' - E_3) \qquad [16\cdot 24]$$

Solution.

$$E' = \frac{E_1C_1 + E_2C_2 + E_3C_3}{C_1 + C_2 + C_3} = \frac{\Sigma Q}{\Sigma C} \qquad [16\cdot 25]$$

Again we find the simple result which the superposition principle would indicate. *The final voltage is that which the net absolute displacement Q would require for a single capacitance equivalent to the parallel combination.*

Solutions for ΔQ_1, ΔQ_2, ΔQ_3 are left to the student.

The foregoing are given primarily as examples to show that the solution of circuits with capacitance is a *straightforward Kirchhoff law procedure,* and should be studied from this point of view; the end results are of secondary importance.

It is desirable now to utilize equations 16·19 and 16·25 in the suggested general Σ form to check the earlier statement that the two-capacitor circuit is either a parallel or series connection merely according to the point of view. As a series circuit Fig. 16·9 gives

$$\Delta Q = \frac{\Sigma E}{\Sigma(1/C)} = \frac{E_1 + 0}{(1/C_1) + (1/C_2)} = E_1 \frac{C_1C_2}{C_1 + C_2} \qquad [16\cdot 14']$$

As a parallel circuit Fig. 16·9 gives

$$E' = \frac{\Sigma Q}{\Sigma C} = \frac{C_1E_1 + C_2(0)}{C_1 + C_2} = E_1 \frac{C_1}{C_1 + C_2} \qquad [16\cdot 12']$$

16·12. Energy in Capacitive Circuits. It is particularly instructive to examine the energy changes involved in these examples. In Fig. 16·9*a* the stored energy (equation 16·4) is

$$W = \tfrac{1}{2}C_1E_1^2 + 0$$

After the switch is closed (Fig. 16·9*c*) the energy becomes

$$W' = \tfrac{1}{2}C_1(E_1')^2 + \tfrac{1}{2}C_2(E_2')^2$$

Substituting the value of E_1' and E_2' (equation 16·13):

$$W' = \tfrac{1}{2}C_1\left(\frac{E_1C_1}{C_1 + C_2}\right)^2 + \tfrac{1}{2}C_2\left(\frac{-E_1C_1}{C_1 + C_2}\right)^2$$

$$= \tfrac{1}{2}(C_1 + C_2)\left(\frac{E_1C_1}{C_1 + C_2}\right)^2 = \frac{(E_1C_1)^2}{2(C_1 + C_2)}$$

The stored energy W' after closure of the switch evidently is *not* the same as W before closure of the switch. Let us examine the ratio

$$\frac{W'}{W} = \frac{(E_1C_1)^2}{2(C_1 + C_2)}\,\frac{2}{C_1E_1^2}$$

$$= \frac{C_1}{C_1 + C_2} \qquad [16\cdot 26]$$

Since $C_1 < (C_1 + C_2)$ it follows that $W' < W$.

This indicates that *some of the original energy W has disappeared!* Observe that this is evidenced in spite of the fact that no means for energy dissipation has been postulated in the idealized circuit of the example. Actually of course there will be some resistance and the energy may be dissipated in i^2R heat. The circuit also is likely to have inductance and it is entirely possible that the displacement ΔQ may at first "overshoot" by reason of the inertia effect of the inductance and then repeatedly reverse or oscillate with decreasing amplitude until the proper ΔQ can be achieved. The oscillating circuit will radiate more or less energy in the fashion familiar to radio. Study of this phenomenon and the relation of energy radiated to energy dissipated in i^2R heat is beyond the scope of this text.

It is important to keep this energy dissipation characteristic in mind for circuits of this nature because there is some temptation to attempt to solve the circuits by applying the principle of conservation of energy. While the principle applies well enough, there is commonly no easy way of determining the dissipated energy, and to assume it zero by postulating no dissipation of energy is here an inexcusable and *major error.* This energy behavior is characteristic of self-contained energy systems of *any kind* in action. Any system of exclusively stored energies can function only to *run down.* In hydraulics, for example, this occurs to the energy of two tanks of different head when connected to permit interflow. In mechanics two springs with different opposed torques (or forces) act to transfer energy from one to the other but always dissipate an amount independent of the mechanism provided for receiving it.

The manner in which this takes place for the capacitor may best be studied by means of the EQ graph previously introduced (Fig.16·8).

The graph of Fig. 16·12b applies to the case where two equal capacitors, $C_1 = C_2$, are connected as in Fig. 16·12a after being charged to different potentials $E_1 > E_2$.

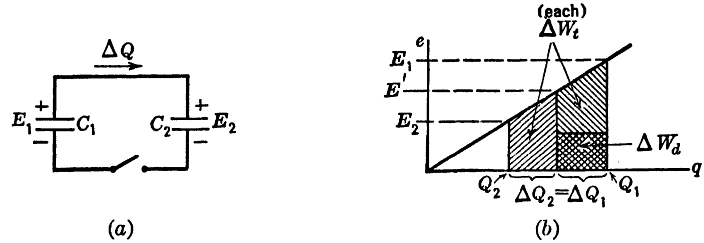


FIG. 16·12.

Following closure of the switch, displacement ΔQ (of equal amount in both capacitors) occurs until voltage E' is established for both capacitors. The areas in the graph show the respective increase and decrease in the energy stored in C_2 and C_1 respectively. Evidently ΔW_t is transferred from C_1 to C_2 while ΔW_d disappears from the capacitors and must be dissipated in some manner as discussed above.

When the capacitors are not of equal value, $C_1 > C_2$, and C_2 is initially uncharged, the graph becomes as in Fig. 16·13a.

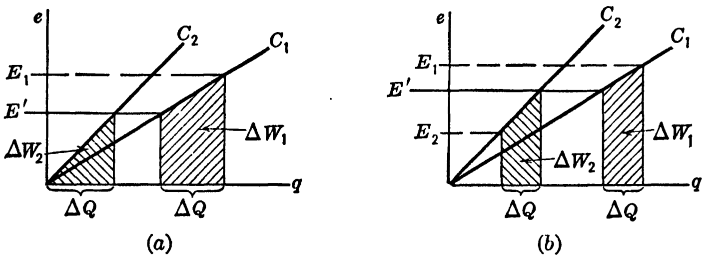


FIG. 16·13.

When C_2 is not initially without charge the graph becomes as in Fig. 16·13b where $E_1 > E_2$. In either (a) or (b) it is clear that $\Delta W_1 > \Delta W_2$ where ΔW_1 is the energy lost by C_1 and ΔW_2 is the energy gained by C_2. The energy dissipated of course is $\Delta W_1 - \Delta W_2$.

While it is entirely possible to extend this graphical representation to the more complicated conditions illustrated in Figs. 16·10 and 16·11, it seems unjustified to extend consideration of the concept here. If the reader wishes to study these, however, it is preferable to shift the C lines

so that the initial values of Q all coincide at the chosen origin, as in Fig. 16·14a. This simplifies the finding of a value E' which will make $\Delta Q_1 = \Delta Q_2$. For the two-capacitor case a graphical solution is best devised by reversing the slope of one of the capacitors as in Fig. 16·14b.

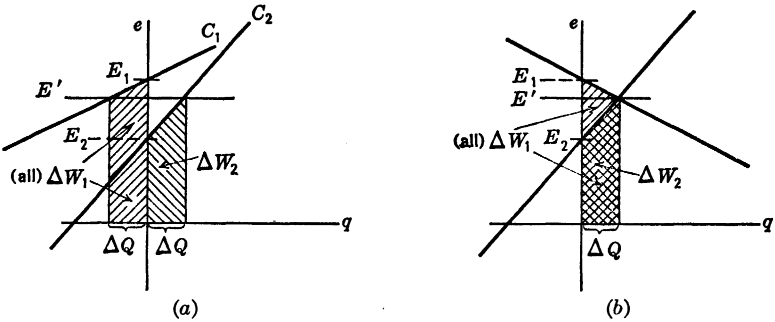


FIG. 16·14.

The intersection of the C lines then establishes the value of E' as well as ΔQ.

16·13. Dielectrics. As previously indicated, the one ingredient of a capacitor which is primarily responsible for its behavior is the dielectric. So long as the "plates" of a capacitor are adequate to conduct the current and to make intimate contact with the dielectric, they impart no character to the capacitor regardless of composition.

There are three major properties of dielectric materials which must be recognized in order to deal intelligently with them and with capacitors.

1. The dielectric constant.
2. The dielectric strength.
3. The dielectric resistivity or leakage.

1. *Dielectric constant.* The principal characteristic of a dielectric is its manifestation of what may be called an *electrical elasticity.* The orbital electrons seem to be elastically restrained or bound to the nuclei of the dielectric as though by elastic bands or leashes. While these *bound* electrons can be displaced from their orbits, the displacement is of *limited* amount and is directly proportional to the "persuasion" of the electric force or potential gradient to which they are subjected.

It has been observed that $Q = CE$ and $C = \kappa(A/l)$. These combine to give

$$Q = \kappa \frac{A}{l} E$$

or

$$\frac{Q}{A} = \kappa \frac{E}{l} \qquad [16 \cdot 27]$$

which we write

$$D = \kappa G \qquad [16 \cdot 28]$$

where D = density of electron displacement in coulombs per unit sectional area.

G = electric potential gradient in volts per unit length of displacement path.

$\kappa = \kappa_s \kappa_0$ = dielectric constant or permittivity in farads per unit length.

Equation 16·27 is rewritten to obtain

$$\kappa = \frac{D}{G} \qquad [16 \cdot 29]$$

We observe that dielectric constant κ is analogous to the mechanical ratio (strain)/(stress) which is the reciprocal of the familiar modulus of elasticity. Thus, the *stress* of electric potential gradient G produces dielectric *strain* of the bound electron structure expressed by D electrons per unit section. This, of course, further indicates the propriety of conceiving dielectric materials to possess *electrical elasticity* and condones the use of the hydraulic analog of Fig. 16·2.

Material	Dielectric Constant κ_s	Dielectric Strength kv/mm	Resistivity ohm-cm
Vulcanized rubber	3.5	10	10^{18}
Transformer oil	2.5	6	10^{15}
Porcelain	5.0	11	10^{15}
Mica	7.5	150	2×10^{17}
Paraffin	2.1	40	5×10^{18}

2. *Dielectric strength.* Just as there is a mechanical stress limit for any given material, there is also a fairly definite limit to the electrical stress which it can endure without permanent "deformation" or actual destruction. The dielectric has a *dielectric strength* which is expressed as a maximum allowable value of the potential gradient G. Values for different materials are tabulated in electrical handbooks as typified in the above table but must be permitted considerable latitude for error when thicknesses other than those tabulated or materials of questionable homogeneity are involved. Beyond these, humidity, temperature, duration of stress, and even the previous operation affect the dielectric

strength of materials. Research has as yet revealed no simple laws or reliable basic data for explaining the breakdown of dielectrics and predicting their strength, although much work has been done and is in progress.

3. *Dielectric resistivity.* For most purposes it is important that dielectric materials be good insulators or nonconductors, that is, *relatively devoid of free electrons.* No dielectric is a perfect insulator, however, and its resistivity bears scrutiny for any given application. Where storage of energy for appreciable time is required, especially in measuring apparatus, the resistivity must be high.

Dielectric leakage or conduction may occur over the *surface* of the material more than *through* the material. The ***surface resistivity*** may

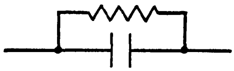


FIG. 16·15.

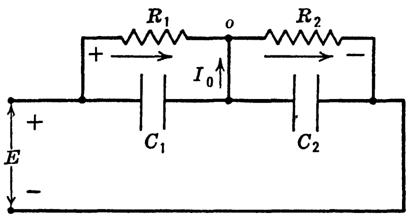


FIG. 16·16.

then be quite as important as the ***volume resistivity.*** Of course only the *volume* resistivity is expressible by the familiar $R = \rho(l/A)$ relation previously studied for conductors.

Both leakages pursue paths in shunt (parallel) with that of the bound electrons and are therefore grouped and represented symbolically by a single resistance in parallel with the capacitor containing the dielectric as in Fig. 16·15.

It will be found of practical importance to observe that the respective voltages of capacitors connected in series is determined by their respective *leakage resistances.* Consider the two capacitors C_1 and C_2 (Fig. 16·16) to be of equal capacitance C. Let the leakage resistance R_1 be twice that of R_2, a not unusual variation for commercial capacitors even of the same make and catalog number. When E is applied, the displacement $Q = \frac{1}{2}CE$ will provide voltages $E_1 = E_2 = \frac{1}{2}E$. Voltage E_2 applied to R_2, however, produces leakage current twice as large as the same voltage E_1 applied to $R_1 = 2R_2$. By Kirchhoff's junction law there must then be a current I_0 in the direction shown which will begin discharging C_2 and charging C_1. This continues until E_1 is raised and E_2 lowered sufficiently to equalize the leakage currents. Where $R_1 = 2R_2$ the voltages of course become $E_1 = 2E_2 = \frac{2}{3}E$. If it has been

presumed that neither capacitor voltage would exceed $\frac{1}{2}E$, a very unpleasant surprise is experienced when the *better* of the two capacitors, C_1 with the higher resistance to leakage, breaks down and allows full voltage E to be applied to C_2 which completes the destruction. Evidently capacitors should be connected in series on direct current *only after checking their leakage resistances* and adjusting the higher ones by suitable values of externally connected resistors. The adjusted resistance of course should be proportional to the voltage rating of the capacitor and is not a function of the capacitance when assorted capacitors are involved. It should be observed further that the leakage resistance of a capacitor is not immune to *aging*, i.e., to large changes as it becomes older.

16·14. Energy Storage in Dielectrics. Returning to equation 16·4, derived for the capacitor, we may substitute $C = \kappa(A/l)$ to obtain

$$W = \tfrac{1}{2}\kappa \frac{A}{l} E^2 = \tfrac{1}{2}\kappa(Al)\left(\frac{E}{l}\right)^2$$

or

$$W = \tfrac{1}{2}\kappa V G^2 \qquad [16\cdot30]$$

Where $\kappa = \kappa_s\kappa_0$, the dielectric constant.
V = volume of the dielectric.
G = potential gradient throughout the dielectric.

It is useful to write this in the form

$$w = \frac{W}{V} = \tfrac{1}{2}\kappa G^2 \qquad [16\cdot31]$$

The energy which can be stored in a unit volume of dielectric then depends *directly* upon the dielectric constant and upon the *square* of the potential gradient.

Clearly, to compute the *maximum* energy storing ability of a dielectric, we use the *square* of the *dielectric strength* as expressed by the *maximum* allowable potential gradient. Special care is necessary to guard against the inconsistent use of units here because the strength gradient is likely to be expressed in volts per mil or kilovolts per millimeter which require conversion to MKS units before computing the energy.

As before noted, the long-time storage of energy involves resistivity or leakage as a *major* factor.

16·15. The Electrostatic Field. For many purposes it is helpful to conceive of electrostatic phenomena in terms of what is called electrostatic or dielectric flux. The flux concept for electrostatics is largely analogous to the flux concept for magnetics. In a solid dielectric the lines of flux coincide with the direction of displacement of the elastically

bound electrons of the dielectric. The number of lines or *magnitude* of the dielectric field ψ is proportional to the total electron displacement $Q = \int i\,dt$, and the flux density $\mathfrak{D}$ is proportional to the density $D = dQ/dA$ of the integrated displacement current in any part of the dielectric.

The following table shows the correspondence between the principal quantities involved in magnetic and dielectric fields. It will be observed that, in the rationalized MKS system, dielectric flux has the same unit of measure as, and numerically equals, electron displacement Q.

Magnetic			Electrostatic		
Unit	**Symbol**	**Quantity**		**Symbol**	**Unit**
Weber	ϕ	Field or flux		ψ	Coulomb
Weber/meter²	B	Flux density (strain)		$\mathfrak{D}$	Coulomb/meter²
Amp-turn (AT)	$\mathfrak{F}$	Potential		E	Volts
Amp-turn/meter	H	Potential gradient (stress)		G	Volts/meter
Weber/Amp-turn	$\mathcal{P}$	Permeance	Capacitance	C	Farad
Weber/Amp-turn meter	μ	Permeability	Dielectric constant	κ	Farad/meter
Weber/Amp-turn meter	μ_0	Absolute permeability	Absolute dielectric constant	κ_0	Farad/meter
Numeric	μ_s	Specific permeability	Specific dielectric constant	κ_s	Numeric

In the same way that the magnetic field has magnetic equipotential lines normal to the lines of flux, so the electrostatic field has electric equipotential lines normal to the lines of electrostatic flux. It follows that the same principles for plotting magnetic fields by the formation of curvilinear squares are applicable to the electrostatic case. Even the analogous tangent relation for refraction of flux holds.

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\kappa_1}{\kappa_2} \tag{16·32}$$

These fields are useful not only in computing the capacitance of irregular shapes but also in observing the flux density and potential gradient throughout the region which is under electrostatic stress. This latter is important because, as before observed, the strength of the dielectric may be expressed in terms of a maximum permissible gradient which is *not* to be exceeded. Prior to actual breakdown the leakage current which produces a glow discharge known as ***corona*** is commonly an important factor also dependent on potential gradient or density of electrostatic

field. Thus the field readily indicates regions of greatest stress and greatest danger of breakdown.

16·16. Fringing—Guard Rings. It may be seen from Fig. 16·17 that the condition at the edges of a sandwich type capacitor may upset the simple $C = \kappa(A/l)$ calculation because A and l are somewhat indefinite. If l is small relative to the dimensions which determine A, the effect of this so-called *fringing* may be negligible for most purposes. For precise measurement of capacitance it is usual to provide what are known as *guard rings* such as GG in Fig. 16·18. The guard rings are

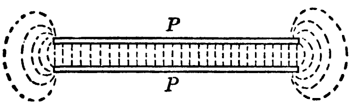


Fig. 16·17.

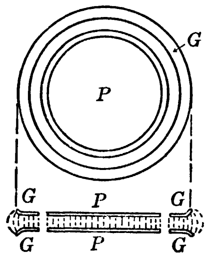


Fig. 16·18.

maintained at the same potentials as the adjacent plates PP by an independent source of voltage. Fringing is thus suppressed for PP and, although still present for GG, is of no consequence there because only PP is involved in the measurement circuit.

It should be observed that in Fig. 16·17 the fringing flux has high density at the electrode edges. This invites breakdown at these edges which may constitute a serious problem. Suitable curvature of the edges may be provided as in Fig. 16·18 to reduce the flux density considerably, however.

16·17. Graded Insulation. The field pattern for a single conductor sheathed cable is shown in Fig. 16·19. It is evident that the dielectric stress is greatest at the conductor C and also that efficient use of the dielectric is not realized by this nonuniform stress. Sometimes one or more intermediate sheaths such as S' in Fig. 16·20 are introduced and maintained at such potential as will make the greatest dielectric stress in each section (CS' and $S'S$) equal or, if different dielectric materials are used, proportional to their respective dielectric strengths. Fig. 16·20 shows that the inner sheath for the illustrated case permits *double* the voltage between *outer* sheath and conductor as compared with Fig. 16·19, i.e., twelve instead of six equal voltage steps (circles).

The general principle illustrated by the sheath S' is finding increasing application. A very common example is found in the so-called *condenser*

type bushing which is used to bring high-voltage conductors through the grounded case of transformers and like equipment without excessive electrical stress between the conductor and the edge of the hole in the

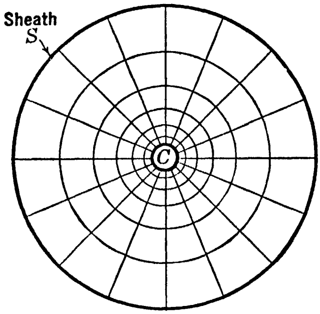


FIG. 16·19.

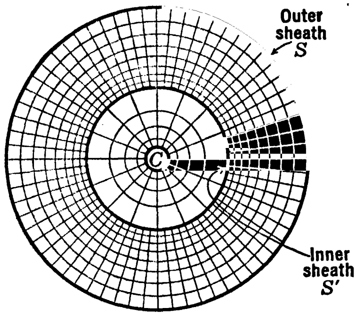


FIG. 16·20.

case. This is illustrated in Fig. 16·21 which shows how the insulation is subdivided into sections *ABC* so proportioned for length and diameter that each section carries equal share of the electrical stress because the capacitance of each section is the same.

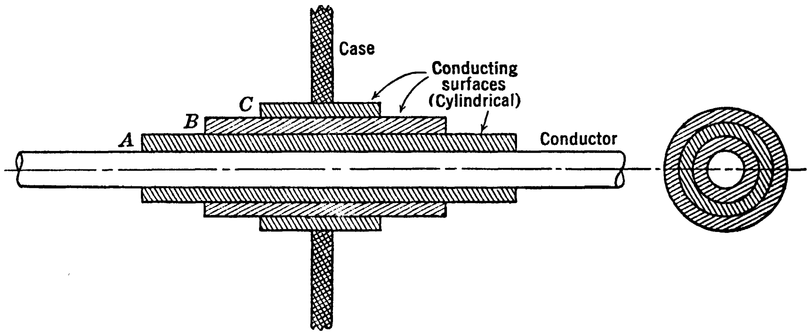


FIG. 16·21. Condenser type bushing.

16·18. Combating Corona. For aerial conductors subject to high voltage as in the Boulder Dam transmission line, the electrical stress is reduced by using a hollow conductor. For a given cross section it is evident that the outer diameter of the hollow conductor is greater than the solid conductor. Even brief consideration of the field mapping for this indicates that lower stress accompanies greater radius of curvature. The lower stress is required in this instance to prevent excessive corona which constitutes an appreciable power loss entirely apart from the usual i^2R loss in the conductor itself.

The construction of the Boulder Dam conductor is an interesting example of ingenuity in providing flexibility for the hollow conductor. This is shown (about 5/6 actual size) in Fig. 16·22.

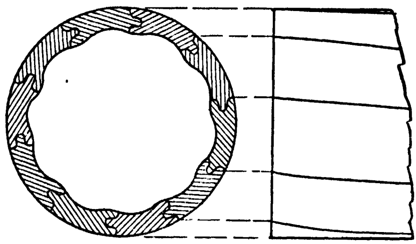


FIG. 16·22. Boulder Dam conductor.

16·19. Energy in the Flux. In the same manner as for magnetics, we conceive that the energy, which we have seen to be stored in a charged capacitor, resides in the field. We have found that this energy may be expressed by $W = \frac{1}{2}CE^2$, $W = \frac{1}{2}QE$ or $W = Q^2/2C$. In terms of dielectric flux, we need only to substitute ψ for Q to obtain

$$W = \tfrac{1}{2}\psi E \qquad [16\cdot33]$$

or

$$W = \frac{\psi^2}{2C} \qquad [16\cdot34]$$

These are clearly analogous to the magnetic $W = \mathfrak{F}\phi/2$ and $W = \phi^2/2\mathcal{P}$.

It should also be observed that equations 16·30 and 16·31 apply to the field concept without alteration. By substituting $\mathfrak{D} = \kappa G$ in equation 16·31 we obtain two other forms for the *energy in a unit volume* of the dielectric field.

$$w = \tfrac{1}{2}\mathfrak{D}G \qquad [16\cdot35]$$

$$w = \frac{\mathfrak{D}^2}{2\kappa} \qquad [16\cdot36]$$

16·20. Electrostatic Force. The electrostatic-magnetic analogy extends to include relations for mechanical forces. Again we utilize the *principle of virtual displacement* to obtain the derivation.

Let us consider a uniform field (no fringing) between the plates of a simple air capacitor connected as in Fig. 16·23*a*. By application of voltage E an amount of energy given by equation 16·34 is stored in the capacitor field.

Now let the source of emf be disconnected and the two plates separated farther as in Fig. 16·23*b*. Since there is no electrical connection to the plates there is no opportunity for electron flow and Q or ψ cannot change.

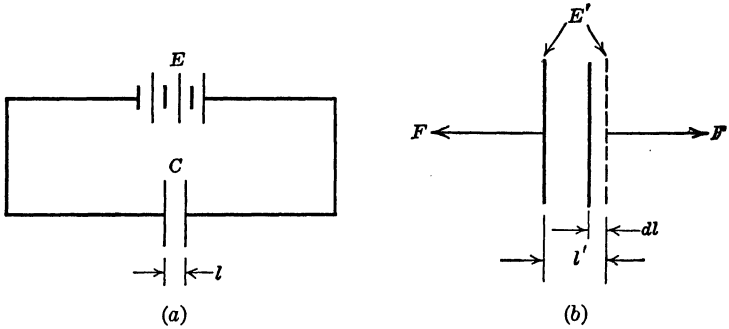


FIG. 16·23.

The capacitance, however, is decreased by increase of l in $C = \kappa(A/l)$ and equation 16·34 gives

$$W = \frac{\psi^2 l}{2\kappa_0 A} \qquad [16\cdot37]$$

It follows from equation 16·37 that the stored energy is increased by the displacement dl in Fig. 16·23*b* as given by

$$dW = \frac{\psi^2}{2\kappa_0 A}\, dl \qquad [16\cdot38]$$

Since the capacitor is disconnected from the electric circuit, the increased energy can come only from mechanical work done on the capacitor during the displacement dl. The force then is

$$F = \frac{dW}{dl} = \frac{\psi^2}{2\kappa_0 A} \text{ newtons (MKS units)} \qquad [16\cdot39]$$

In terms of field density this becomes

$$F = \frac{(\mathfrak{D}A)^2}{2\kappa_0 A} = \frac{\mathfrak{D}^2 A}{2\kappa_0} \qquad [16\cdot40]$$

The force per unit area or *pressure* is

$$\frac{F}{A} = \frac{\mathfrak{D}^2}{2\kappa_0} \qquad [16\cdot41]$$

This is the same value that equation 16·36 gives with air and, as in magnetics, we find that the ***force per unit area is simply equal to the density of the field energy.***

16·21. Transverse Force. If each plate of the capacitor of Fig. 16·23 be made of two overlapping thin sheets the effect of changing the plate area (and consequently the sectional area of the field) is readily observed. The change in energy for a reduction in area dA is, from equation 16·37,

$$dW = \frac{\psi^2 l}{2\kappa_0}\left(-\frac{1}{A^2}\right) dA \qquad [16\cdot 42]$$

Since *reduction* in area is expressed by $dA = -a\,ds$ (with minus sign), where a and s are the plate dimensions, the force must be:

$$F = \frac{dW}{ds} = -a\frac{dW}{dA} = \frac{a\psi^2 l}{2\kappa_0 A^2} \qquad [16\cdot 43]$$

The pressure, or force per unit area of *field*, taken normal to the *displacement* (not normal to A), is:

$$\frac{F}{al} = \frac{\psi^2}{2\kappa_0 A^2} = \frac{\mathfrak{D}^2}{2\kappa_0} \qquad [16\cdot 44]$$

Again as for the magnetic field, the transverse and longitudinal pressures have the same magnitude, the one in tension and the other in compression. Furthermore it is to be noted that either of these pressures tends to *in*crease the capacitance, and we state that ***the mechanical forces produced by electrostatic fields are in such direction as will tend to change the conformation of the field into one of higher capacitance.***

16·22. The Condenser Microphone. It is to be observed in each of the foregoing, where flux ψ is constant, that the potential between the plates increases with increase in stored energy. This is immediately evident from equation 16·33. Furthermore it is readily shown that the emf is directly proportional to the *mechanical displacement.* For the first case (Fig. 16·23) where one plate was pulled away from the other by amount dl, the relation is

$$\psi = CE = \kappa\frac{A}{l}E$$

$$E = \left(\frac{\psi}{\kappa A}\right) l \qquad [16\cdot 45]$$

or

$$dE = \left(\frac{\psi}{\kappa A}\right) dl \qquad [16\cdot 46]$$

The apparatus thus functions as a *generator of emf.* Provided no current flow is permitted, the emf is a replica of the mechanical displacement of the plate and may be used as a "condenser microphone" or as a vibration pickup for phonographs, musical instruments, machinery, etc. The requirement that no current flow is readily approximated by utilizing the emf only to operate the suitably designed grid control of an amplifier tube of conventional radio type.

It is to be noted that this device differs basically from the corresponding electromagnetic device previously studied in that the ***electrostatic emf is proportional to the displacement while the electromagnetic emf is proportional to the velocity of the moving element,*** as shown by equation 16·45 and by the familiar:

$$e = (Bl)v$$

This distinction dictates basic differences in the mechanical requirements for the two types which cannot be ignored in microphone or reproducer design, or in the use of vibration pickups.

16·23. Mechanical Displacement with Constant Potential. All the above cases have concerned displacement with the capacitor effectively *disconnected* from an electric circuit and therefore with constant Q and ψ. If the displacement occurs with a constant potential or battery supply *continuously connected* as in Fig. 16·23*a*, the consequences are uniquely different and worthy of special note. Let us separate the plates amount dl as before. The reduction in capacitance C, with E constant, will cause an electron displacement.

$$dq = E\,dC \tag{16·47}$$

In contrast with the preceding case for constant Q, observe clearly that the reduction of capacitance for constant E is accompanied by *reduction* in stored energy, as shown by

$$W_C = \tfrac{1}{2}CE^2$$

$$dW_C = \tfrac{1}{2}E^2dC \tag{16·48}$$

The direction of dq for *dis*charging the capacitor, as required by the above, will be such as to *charge* the battery by an amount (with constant E):

$$dW_b = E\,dq \tag{16·49}$$

Substituting equation 16·47 in 16·49 gives

$$dW_b = E^2dC \tag{16·50}$$

Now observe carefully that equations 16·48 and 16·50 indicate that the battery receives *twice* as much energy as that *released* from the

capacitor. The discrepancy of course is accounted for by the mechanical work dW_m done in displacing the capacitor plates and we write

$$dW_b = dW_C + dW_m$$

$$dW_m = dW_b - dW_C = \tfrac{1}{2}E^2dC$$

or

$$dW_m = dW_C \qquad [16 \cdot 51]$$

In other words, for each joule of *mechanical* energy converted and contributed to the battery, the *capacitor* matches the contribution with another joule to make 2 joules total to the battery.

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QUESTIONS

16·1. Define capacitance. Define capacitor.

16·2. Give the unit of measure of capacitance and show how it is related to other quantities concerned.

16·3. Expound an hydraulic analog for the capacitor.

16·4. What is the function of the plates of a mica capacitor?

16·5. Expound the distinction between κ_s and κ_0 for dielectrics.

16·6. Derive three relations for the energy stored in a capacitor.

16·7. What property of two-element circuits, such as Fig. 16·9, places them in a special classification as compared with circuits having more than two elements?

16·8. Explain why the polarity assumed for the final voltage of each capacitor in Fig. 16·10*b* is taken to be the same as the initial polarity (Fig. 16·10*a*) although it is certain that all these cannot be the true polarities.

16·9. Explain how it is known that equation 16·16 should be as shown and not with the right side $C_1(E_1 - E_1')$.

16·10. Draw a graph like Fig. 16·13*b* for $C_2 > C_1$ and $E_1 > E_2$.

16·11. Draw a graph like Fig. 16·13*b* for $C_1 > C_2$, $E_1 > E_2$, and with the *reverse* initial polarity shown for E_2 in Fig. 16·12.

16·12. Explain why the conservation of energy principle cannot be used to solve electric circuits containing capacitance alone.

16·13. Two capacitors of commercial quality (not ideal), each tested and rated for the same capacitance and voltage, are connected in series across a d-c voltage somewhat less than twice the rated voltage of either capacitor. Explain why one

(and probably both) may be expected to break down. What can be done to safeguard the operation of capacitors in this manner?

16·14. Show an electric circuit with resistance and capacitance which takes account of the effects of dielectric leakage.

16·15. Mica used in an electric toaster is referred to as an *insulator* while mica used in a capacitor is termed a *dielectric*. Explain the distinction in terms.

16·16. Distinguish between *free* and *bound* electrons and make clear the nature and extent of the *freedom* and *bonding* of each, respectively.

16·17. (*a*) Derive a relation to show what factors determine the maximum energy-storing ability of a dielectric.

(*b*) Which of these factors is most influential?

16·18. Electric energy is represented by $dW = e\,dq$. Derive an expression for the energy stored in a dielectric (capacitor) in terms of E and Q.

16·19. The energy stored in a storage battery, assuming constant voltage E and negligible resistance, is $W = EQ$ as derived from $dW = e\,dq$. Explain in terms of physical principles (analogy if desired) why the energy stored in a capacitor $W \neq EQ$.

16·20. Sketch and describe the use of a *guard ring* for dielectric measurement, explaining clearly the purpose and the principles involved.

16·21. Derive a relation to show that the mechanical force, which tends to pull the two plates of the simple capacitor of Fig. 16·18 together, is directly proportional to the square of the potential gradient between the plates, and to the dielectric constant of the dielectric between the plates.

16·22. Sketch the map of dielectric flux and equipotential lines for a two-conductor power line.

16·23. In what important respect does the condenser microphone differ fundamentally from the ribbon (electromagnetic) microphone as regards the relation of its generated emf to the mechanical source?

PROBLEMS

16·1. Two capacitors, A of 100 μf capacitance and B of 200 μf capacitance are connected in series across 120-volt d-c mains. Find the energy in each capacitor.

16·2. A multiple plate variable air capacitor has 20 positive plates and 19 negative plates which are uniformly separated by $\frac{3}{32}$ in. of air. The plates are half sections of a circle whose diameter is 4 in.

(*a*) Find the maximum capacitance, assuming that there is no leakage flux.

(*b*) If the whole capacitor is immersed in chlorinated diphenyl what will be the capacitance?

16·3. An air capacitor like Fig. 16·6*b* has cylinders, one with outer diameter $\frac{1}{2}$ in., the other with inner diameter 0.540 in. What is the capacitance when the cylinders overlap $\frac{3}{8}$ in.?

16·4. A telephone capacitor is made up of alternate sheets of 3-mil kraft paper and 2-mil tinfoil. The tinfoil sheets are 4 by 8 in. There are 200 sheets of paper. Alternate sheets of tinfoil are connected together to form one terminal and the remaining sheets form the other terminal. Find the capacitance of the capacitor.

16·5. A rolled paper capacitor like Fig. 16·5 is made with two strips of 3-mil paper 1 in. wide. What total length of paper is required to make a capacitor of 0.10 μf?

16·6. Given the permittivity $\kappa_a = 8.84 \times 10^{-12}$ farad per meter, compute the permittivity of air for the inch cube.

16·7. A No. 10 wire is uniformly coated with a homogeneous dielectric (for insulation) of 1/8 in. thickness and dielectric constant $\kappa_s = 3$. The whole is sheathed in lead for protection against moisture. Compute the capacitance between wire and sheath per foot of wire. (Note that the structure may be analyzed as an infinite number of capacitors in series, each capacitor comprising dielectric of thickness dr (r = radius) and section $A = 2\pi r$; that the solution involves the integral calculus.)

16·8. Capacitor A contains 150 microjoules at 320 volts. When connected to an uncharged capacitor B through a 5-megohm resistor the energy of A is ultimately reduced to 60 microjoules.

(*a*) What energy does capacitor B receive?

(*b*) What maximum heat power does the resistor deliver?

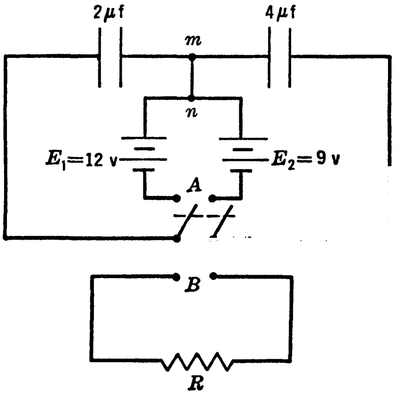
16·9. A capacitor of 20 μf capacitance which has been charged to a potential difference of 220 volts and a capacitor of 10 μf capacitance which has been charged to a potential difference of 110 volts have their positive terminals connected.

(*a*) What is the potential difference between their negative terminals?

(*b*) What is the potential across each capacitor after the negative terminals are connected?

(*c*) What energy has each capacitor before and after the negative terminals are connected?

16·10. The two capacitors are initially without charge. The double-pole, double-throw switch is then closed in position A.



(*a*) How much energy is taken from each battery?

(*b*) What is the potential across each capacitor after the switch is moved to position B and current ceases flowing?

(*c*) If the initial rate of energy flow to R is 60 watts compute the initial current through R.

(*d*) How many joules are delivered to R?

16·11. Same as Problem 10 except link mn removed.

16·12. Three capacitors with capacitances of 10, 20, and 25 μf are charged in series across 220-volt mains and are then disconnected from each other and from the mains without loss of charge. If the capacitors are arranged in parallel with their terminals of like sign connected together, what will be the final potential across them?

16·13. Four capacitors with capacitances of 5, 10, 20, and 25 μf respectively are charged in series across 220-volt mains. They are disconnected without loss of

charge and joined up in parallel, with the negative terminals of the first two capacitors connected to the positive terminals of the others, and the positive terminals of the first two to the negative terminals of the others.

(a) What will be the initial and final Q for each capacitor?

(b) What will be the final potential across the capacitors?

(c) If the four capacitors had been charged in parallel across 110-volt mains and then connected in series in a complete circuit, the positive terminal of one to the negative terminal of the next, what would have been the final Q for each capacitor? Compare the initial and the final polarity of each capacitor.

16·14. Four metallic plates are stacked sandwich style with intervening high-grade dielectric of uniform and equal thickness. A potential $E = 1200$ volts, applied across the outer plates, is accompanied by a displacement of 2.4 microcoulombs.

(a) While potential E is applied as above, the two inner plates are connected. What displacement Q will occur through the connecting wire? Sketch plates and show polarities.

(b) With potential E still applied, the wire connecting the inner plates is removed. What is then the potential between each pair of adjacent plates? Sketch plates and show polarities.

(c) The source of potential E is now removed and the *outer* plates are connected. What displacement Q occurs through the connecting wire and what is now the potential between each pair of adjacent plates? Sketch plates and show polarities.

(d) For which of the above four situations is the energy stored in the stack *greatest?* How much?

(e) For which of the above four situations is the energy stored in the stack *least?* How much?

16·15. For the materials tabulated on page 298:

(a) Determine the maximum amount of energy that can be stored in a cubic centimeter of the tabulated dielectrics.

(b) Which dielectric may be expected to be best for a long-time storage?

(c) Which property of a dielectric has the greatest influence on its maximum energy-storing ability?

16·16. Two large metal plates are separated 2 mm by an air space and subjected to a potential difference of 10,000 volts. Between these metal plates are inserted two insulating plates, one of glass, dielectric constant of 8.5; the other of mica, dielectric constant of 6.5; each 0.6 mm thick. The surfaces of the metal and of the insulating plates are parallel to each other. What is the potential gradient in

(a) The remaining air space.

(b) The mica plate.

(c) The glass plate?

16·17. A lead-covered cable has a core of No. 4 B&S gage copper wire insulated with rubber. What thickness of rubber insulation must be used so that the maximum electric stress in it will not exceed 150 volts per mil when a potential of 15,000 volts is applied between the core and the sheath?

16·18. (a) A sheath cable of single No. 0 solid conductor is designed for 6600 volts between sheath and conductor (see Fig. 16·19). The maximum allowable gradient for the design, including safety factor, is 3 kv per millimeter. What inside diameter of sheath is required for minimum volume of dielectric per unit length of cable?

(b) If an intermediate sheath such as S', Fig. 16·20, is included for operation at one-half total potential, what per cent of the dielectric used in the first design (a) can be saved? Neglect thickness of sheath S'.

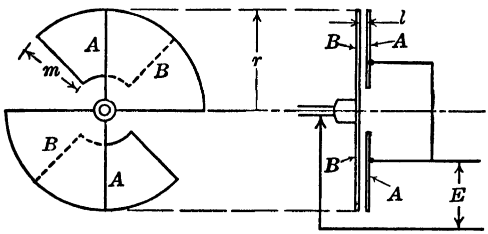
16·19. Two capacitors $C_1 = 500$ $\mu\mu$f and $C_2 = 1000$ $\mu\mu$f each with leakage resistance of 1 megohm are connected in series across a constant potential of 600 volts.

(*a*) How much energy is stored in each capacitor immediately after application of the 600 volts?

(*b*) How much energy is taken from the source of emf during charge (neglect energy loss in leakage resistance)?

(*c*) If the constant voltage is applied for some time (an hour or so) what will be the energy stored in each capacitor?

16·20. The apparatus in the figure consists of four quarters of a metal disk *AABB* so mounted that *BB* may rotate like the blades of a zero-pitch fan in front of the



stationary blades *AA*. Parts *AA* are insulated from *BB* by air gap *l*, and potential *E* is applied as shown with the expectation that electrostatic force will rotate assembly *BB* and constitute an electrostatic motor. Given the following data:

$E = 100{,}000$ volts $\qquad r = 20$ cm (radians)

$l = 4$ mm $\qquad m = 4$ cm

(*a*) Compute the maximum capacitance of the motor, neglecting fringing and stray capacitances of wires, etc.

(*b*) Compute the electrostatic energy stored in the $\frac{1}{2}C_{\text{max}}$ position shown in the figure.

(*c*) Compute the torque in foot-pounds produced in the position shown. This is done by observing that torque is change of energy per unit angle of displacement $T = dW/d\alpha$. Change of capacitance and change of energy dW can be computed for angular displacement $d\alpha$.

(*d*) From the relations involved in (*c*) determine the nature of the function $T = f(\alpha)$ and sketch a graph of T for 360 degrees rotation.

(*e*) Explain what change in operation or design is required to avoid reversal of torque found in (*d*).

(*f*) Discuss the practicability of the motor in some detail and explain clearly what practical difficulties (one especially) would prevent this particular design from working even after the modification in (*e*). Dielectric strength of air is approximately 3 kv per millimeter at atmospheric pressure.

Note. Motors similar to the above have been built and operated to produce $\frac{1}{4}$ hp or more. The necessary refinements of design and construction, as well as lack of satisfactory power supply, prevent its practical use.

CHAPTER XVII

CIRCUITS WITH RESISTANCE AND CAPACITANCE

17·1. Resistance and Capacitance in Series. Our study of capacitance as an electric circuit element so far has been confined to a consideration of conditions immediately *before* and *after* certain switching operations. We are no less concerned with the phenomena from instant to instant *during* the period over which these operations are taking place.

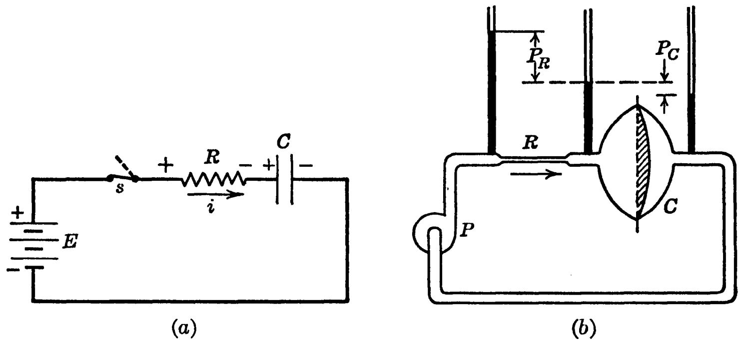


FIG. 17·1. (*a*) A series *RC* circuit during application of constant voltage. (*b*) Hydraulic analog for a series *RC* circuit during charge.

Let us consider the circuit of Fig. 17·1*a* which represents the simple combination of a capacitor in series with a resistor. The hydraulic analog is given in Fig. 17·1*b*.

After the closure of switch *s* Kirchhoff's voltage relation gives

$$E - e_R - e_C = 0 \qquad [17\cdot1]$$

Substituting,

$$E = e_R + e_C = Ri + \frac{q}{C} \qquad [17\cdot2]$$

Taking the derivative with respect to *t*,

$$0 = R\frac{di}{dt} + \frac{1}{C}\left(\frac{dq}{dt} = i\right)$$

$$R\frac{di}{dt} + \frac{i}{C} = 0 \qquad [17\cdot3]$$

This is a well-known differential equation of the simplest kind. It has already been encountered and solved in dealing with R and L in Chapter XV.

17·2. Solution of the Equation. We write

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating

$$\ln i = -\frac{t}{RC} + \ln K$$

$$i = K\epsilon^{-\frac{t}{RC}} \qquad [17\cdot4]$$

To find the constant of integration K we observe that when $t = 0$, $q = 0$ and equation 17·2 yields:

$$E = Ri + \frac{0}{C}$$

or

$$i = \frac{E}{R} \qquad [17\cdot5]$$

Substituting equation 17·5 in 17·4 gives

$$\frac{E}{R} = K\epsilon^{-\frac{0}{RC}} = K$$

Equation 17·4 then becomes

$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}} \qquad [17\cdot6]$$

It follows that

$$e_R = Ri = E\epsilon^{-\frac{t}{RC}} \qquad [17\cdot7]$$

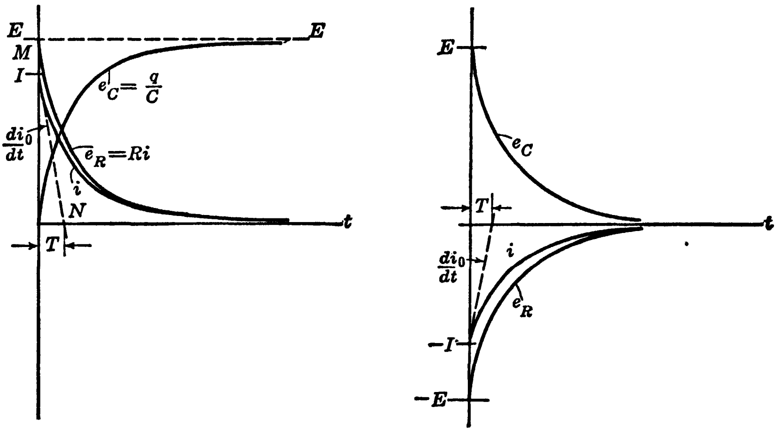
$$e_C = E - e_R = E\left(1 - \epsilon^{-\frac{t}{RC}}\right) \qquad [17\cdot8]$$

The curves for i, e_R, and e_C are given in Fig. 17·2*a*.

17·3. Time Constant. The series circuit of R and C is characterized by a time constant of the same nature as that for the series circuit of R and L. In fact, all the characteristics of the exponential function discussed in Chapter XV for the RL circuit apply to the RC circuit and to many other phenomena of electrical and of nonelectrical nature.

Time constant T is the time that would be required for the current to reach ultimate value zero (or the capacitor to be fully charged) if it continued to decrease at the initial rate indicated by M N in Fig. 17·2a. This initial rate is given by equation 17·3 where i has initial value I_0:

$$\frac{di_0}{dt} = -\frac{I_0}{RC} \tag{17·9}$$



(a) (b)

FIG. 17·2.

For the current to change from I_0 to zero at the *constant* rate di_0/dt of the time constant concept requires time given by T in

$$\frac{di_0}{dt} = -\frac{I_0}{T} \tag{17·10}$$

From equations 17·9 and 17·10 we find that

$$T = RC \tag{17·11}$$

Although the above concept is to be preferred in general, the time constant may also be expressed as the time required for t/RC to become unity in equation 17·6. Obviously $t/RC = 1$ when $t = T = RC$. Numerically this is the time required for i to decrease to $1/2.718 = 0.368$ of the initial value $I_0 = E/R$.

It should be noted, as for the RL circuit, that T is also *at any instant* the time required for the current to reach its goal (zero is the goal here) if it could *continue* at the rate di/dt of that instant (see Fig. 15·10).

17·4. Discharging the Capacitor. When the emf E of Fig. 17·1a is removed, after fully charging the capacitor, and the circuit is closed as in Fig. 17·3a, equation 17·1 becomes

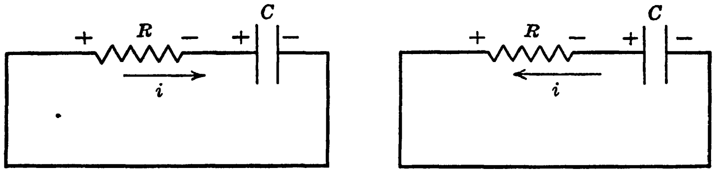
$$0 - e_R - e_C = 0$$

or

$$Ri + \frac{q}{C} = 0 \qquad [17\cdot12]$$

and

$$R\frac{di}{dt} + \frac{i}{C} = 0 \qquad [17\cdot13]$$



(a) (b)

FIG. 17·3.

This last is the same as equation 17·3 for which we found the solution

$$i = K\epsilon^{-\frac{t}{RC}} \qquad [17\cdot14]$$

Here, however, when $t = 0$, the charged capacitor has $q = CE$. Substituting this in equation 17·12 gives

$$Ri + \frac{CE}{C} = 0$$

or

$$i = -\frac{E}{R} \qquad [17\cdot15]$$

Substituting 17·15 in 17·14 to find and eliminate K finally gives

$$i = -\frac{E}{R}\epsilon^{-\frac{t}{RC}} \qquad [17\cdot16]$$

Also we find

$$e_R = Ri = -E\epsilon^{-\frac{t}{RC}} \qquad [17\cdot17]$$

and

$$e_C = 0 - e_R = +E\epsilon^{-\frac{t}{RC}} \qquad [17\cdot18]$$

The curves for i, e_R, and e_C are given in Fig. 17·2*b*. Clearly here the current curve is of the same *shape* as in Fig. 17·2*a* but flow is in the opposite direction, as shown in Fig. 17·3*b*. Observe that there is *no reversal of polarity* for the capacitor voltage on discharge as compared with that during charge.

17·5. The Relative Effects of *R* and *C*. For the series combination of R and L it was observed from the time constant $T = L/R$ that more inductance entails more time for the current growth (or decay) but that more *resistance* involves *less* time for the current growth (or decay). The effect of the inertia-like inductance seems obvious but the inverse effect of the resistance is less readily rationalized. There are two factors which indicate why increased resistance effects decreased time of growth.

1. Increase of resistance decreases the relative influence of the inductance.

2. Increase of resistance decreases the steady-state value of current and consequently the amount of change which is to occur in a given case.

For the series combination of R and C we observe (equation 17·11) that the effect of R is the same as that of C on the time constant. In contrast with the inductive case, *more R* means *more time* instead of less for the current change. This is readily justified because, with the given applied voltage E, a certain displacement $Q = CE$ is indicated for the given capacitor. Higher resistance slows up the rate of this displacement so that more time is required for its completion. The effect of C is readily understood because, for a given voltage E, more C requires more displacement Q and more time to accomplish it by flow through a given fixed resistance.

17·6. Incomplete Growth and Decay. When a capacitor in series with a resistor is alternately charged and discharged in the same manner as was done for the inductance in Fig. 15·5, the *voltage* across the capacitor varies in the same manner as did the *current* through the inductance in Fig. 15·9. The same story in general applies to capacitance and inductance provided the current and voltage are interchanged. This kind of relation might well be expected from the nature of the basic relations

$$e = L\frac{di}{dt} \qquad i = C\frac{de}{dt}$$

which are mathematically the same except for interchange of e and i when C is substituted for L. The reader should develop for e the general equation corresponding to equation 15·24, page 257.

Unlike the case for inductance it is unnecessary to avoid opening the circuit during switching operations and it is feasible to switch from E to a zero resistance substitute as in Fig. 17·4*a*.

This makes possible the same time constant for discharge as for charge, if desired, for which Figures 17·4*b* and *c* show respectively the capacitor voltage and the current.

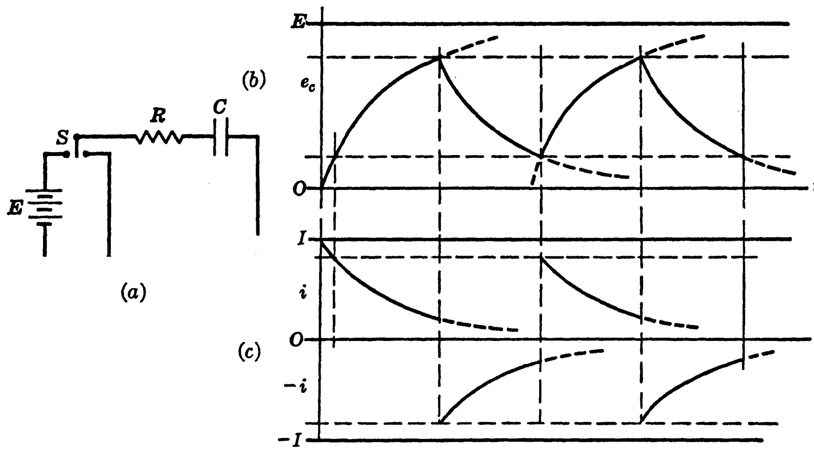


FIG. 17·4.

In slightly different form a very practical application of the periodically switched *RC* circuit is commonly used to provide the sawtooth-shaped sweep voltage for the cathode-ray oscillograph.

The skeleton circuit (Fig. 17·5) consists of a constant source of emf *E* applied to a series *RC* combination. Switch *S* is a gas-filled tube which, like the neon glow lamp of Problem 15·2, takes no current until its ignition voltage is reached. By making *E* several times this ignition voltage, the length (*ab*, Fig. 17·6) of the exponential e_C curve, which is traced before ignition (and the equivalent closure of switch *S*) is so short as to

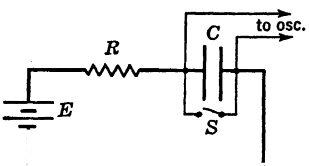


FIG. 17·5.

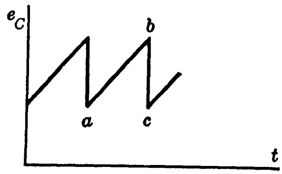


FIG. 17·6.

approximate a straight line. The tube resistance, or resistance of switch *S*, is so small relative to *R* that the time constant for discharge is near zero and produces *bc* in Fig. 17·6 nearly vertical.

When more than usual precision for linear sweep is required it is possible to introduce into the circuit an electron tube of standard radio type

which will control the current through R to constant value. So long as the capacitor has negligible leakage and constant capacitance, constant current will insure a straight line for ab in Fig. 17·6.

The neon lamp previously mentioned was actually used in the pioneer development of this sweep circuit. Because the lamp extinguishes at a voltage but little below its ignition voltage, the capacitor is by no means completely discharged when switch S opens. For a given lamp (or tube) and potential E, there are limits to the range of R and C which will produce oscillation. Within these limits reduction of either R or C will increase the frequency of oscillation and constitute the means for the essential sweep frequency control of the oscillograph. A combination of stepped capacitance and vernier resistance control is usual.

17·7. Power and Energy for Series *RC*. The power taken by either R or C in series during charge from a constant potential is readily found from the usual $p = ei$. For the resistance the modified form $p_R = Ri^2$ together with equation 17·6 gives

$$p_R = R\left(\frac{E}{R}\epsilon^{-\frac{t}{RC}}\right)^2 = \frac{E^2}{R}\epsilon^{-\frac{2t}{RC}} \qquad [17\cdot19]$$

The greatest value of this power, like the current, occurs at the initial instant $t = 0$ and, of course, has value

$$(P_R)_{\max} = \frac{E^2}{R} \qquad [17\cdot20]$$

The energy taken by the resistance during time $t = \infty$, required for complete charge of the capacitor, is

$$W_R = \int p_R\,dt = \frac{E^2}{R}\int_0^\infty \epsilon^{-\frac{2t}{RC}}\,dt$$

$$= \frac{E^2}{R}\left(-\frac{RC}{2}\right)(\epsilon^{-\infty} - \epsilon^{-0})$$

$$W_R = \tfrac{1}{2}CE^2 \qquad [17\cdot21]$$

For the energy stored in the capacitor, we already know from equation 16·4, without integration here, that

$$W_C = \tfrac{1}{2}CE^2 \qquad [16\cdot4]$$

Evidently we have here again the peculiar 50–50 energy relation encountered in Chapter XVI. ***Regardless of the value of either R or C, the energy supplied to them in series from a constant potential supply is evenly divided between them.***

The power taken by the capacitor is given by equations 17·6 and 17·8

$$p_C = e_C i = E\left(1 - \epsilon^{-\frac{t}{RC}}\right)\frac{E}{R}\epsilon^{-\frac{t}{RC}}$$

$$= \frac{E^2}{R}\left(\epsilon^{-\frac{t}{RC}} - \epsilon^{-\frac{2t}{RC}}\right) \qquad [17\cdot22]$$

Since the voltage is zero when $t = 0$, and the current is zero when $t = \infty$, the power product is very likely to have a maximum value at some finite time t. It is left for the student to perform the simple differentiation involved in finding this maximum and the time at which it occurs. The maximum value is independent of C but the time is not.

Observe that, on *discharge* (Fig. 17·2*b*), e_C and i both have their greatest value at $t = 0$ and therefore the capacitor must deliver maximum power at the initial instant of discharge. The value is readily found to be

$$(P_C)_{\max} = E\frac{E}{R} = \frac{E^2}{R} \qquad [17\cdot23]$$

where E is the initial value of e_C, and R is the circuit resistance during discharge. If R is made sufficiently small the power may be many times that available for *charging* the capacitor.

A noteworthy application of this ready source of brief high power is found in the spot welding of small parts such as the filament and lead wires of incandescent lamps and radio tubes. Because the amount of energy for each weld is accurately *metered out* by the $W = \frac{1}{2}CE^2$, a high degree of uniformity is obtainable in high-speed production without elaborate control apparatus.

This method is now being applied to the spot welding of sheet metal in sizes involving a power supply of 50 kva or more. It has been found particularly advantageous for welding metals of high thermal conductivity, such as aluminum, because especially high currents of brief duration are required to complete the weld before the heat can be conducted away from the joint. Charging currents of the order of 1000 amp at some 150 volts from rectified alternating current are being used.

17·8. Capacitance in A-C Circuits. While it is usual to introduce capacitance through the functional relation $q = Ce$ it soon becomes evident in dealing with electric circuits that the derivative form $i = C(de/dt)$, for *constant* capacitance, more clearly expresses its function for this restricted case. From this relation we readily appreciate that capacitance becomes manifest by a *change* in the voltage across it.

Current flows through a constant capacitance only when its voltage is changing. For this reason capacitance, like inductance, is of major importance in a-c circuits and merits attention equal to that given inductance.

When $e = f(t)$ is known analytically the finding of i is not difficult. If $e = f(t)$ is known only as a graph, i may still be found by plotting the slope of the curve. When $i = f(t)$ is known analytically, it may or may not be possible to find e by integration of $e = \frac{1}{C}\int i\,dt$, but it is always possible closely to approximate the graphical representation of $e = f(t)$ by measuring the area under the i-t curve.

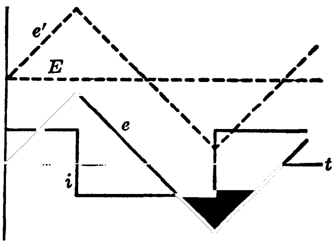


FIG. 17·7.

A comparison of the capacitance relation $i = C(de/dt)$ and the inductance relation $e = L(di/dt)$, as before noted, indicates that they are mathematically the same, but with e and i interchanged. It follows that what has been learned about e in relation to i for inductance is here applicable to i in relation to e for capacitance. Figure 17·7 shows the current i for a triangular wave of voltage e, and should be compared with Fig. 15·12 together with the accompanying discussion.

17·9. The D-C Component. The d-c component of voltage is of no less practical importance for capacitance than is the d-c component of current for inductance. One very common application is illustrated in Fig. 17·8, the schematic circuit for a vacuum-tube amplifier stage. Tube T and battery B produce e' which, of course, is a composite of alternating and direct voltage superimposed. Since the d-c component of e' cannot produce continuous current through capacitor C, voltage e across the second resistor is pure alternating current like that produced by T. Capacitors used in this way are known as ***blocking condensers*** because they block the d-c component of voltage which is unavoidable in T-B but unwanted at e.

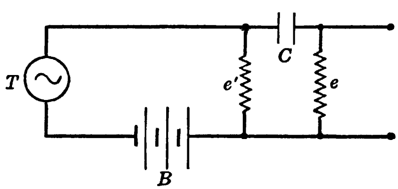


FIG. 17·8.

17·10. For Sinusoidal Circuits. When the voltage across a constant capacitance is $e = E_m \sin \omega t$, the current $i = C(de/dt)$ is readily found analytically.

$$i = C\omega E_m \cos \omega t = \omega C E_m \sin\left(\omega t + \frac{\pi}{2}\right) \qquad [17\cdot 24]$$

or

$$i = I_m \sin\left(\omega t + \frac{\pi}{2}\right) \qquad [17 \cdot 25]$$

where

$$I_m = \omega C E_m \qquad [17 \cdot 26]$$

Again as for inductance, we make three important observations.

1. *Sinusoidal* emf across capacitance is accompanied by *sinusoidal* current through the capacitance (equation 17·25).

2. The *current* leads the emf by 90 degrees, i.e., is in leading quadrature with the emf. It is equally true that the emf lags the current by 90 degrees (equation 17·25) (see Fig. 17·9).

3. The amplitude of the *current* is directly proportional, not only to the emf amplitude and to the capacitance, but also to the *frequency* ($\omega = 2\pi f$ in equation 17·26).

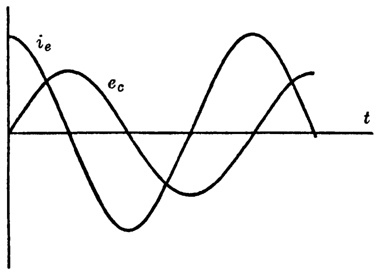


FIG. 17·9. Sinusoidal voltage and current for a capacitor.

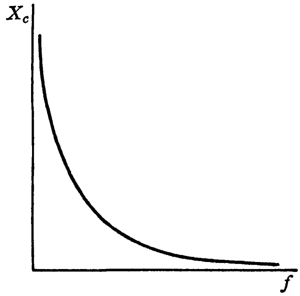


FIG. 17·10. Variation of X_C versus f for one value of C.

17·11. Capacitive Reactance. From equation 17·26 we write

$$\frac{E_m}{I_m} = \frac{1}{\omega C} \qquad [17 \cdot 27]$$

and

$$X_C = \frac{1}{\omega C} \qquad [17 \cdot 28]$$

The quantity X_C is measured in ohms like X_L and is called ***capacitive reactance.***

Like inductive reactance it differs from resistance in the same four major respects (cf. p. 263).

1. Although reactance is not itself a sinusoidal quantity its conception arises out of sinusoidal emf and current. It is meaningless for nonsinusoidal emf and current.

2. Reactance is a function of *time* or frequency. The specification of a reactor must include frequency as well as reactance (ohms) and current.
3. Reactance is defined by the ratio $X = E_m/I_m$ but, unlike R, is *not* also the ratio e/i where e and i are respectively the voltage across and current through the reactance at any instant.
4. Reactance dissipates no energy in heat.

Item 1, as mentioned for inductance, is commonly violated in some degree and called *effective reactance* for waves which only approximate the sinusoidal. The risk in so doing is no less for capacitive than for inductive reactance.

Item 2 is represented graphically in Fig. 17·10 and analytically by equation 17·28 which expands to:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \left(\frac{1}{2\pi C}\right)\frac{1}{f} \qquad [17\cdot29]$$

This frequency characteristic of capacitive reactance is of major importance and is directly concerned with most of the a-c applications of capacitance. Note clearly that it has the *inverse* frequency characteristic of inductive reactance. ***Capacitive reactance is inversely proportional to frequency.***

In the same sense that we have written the *constructional relations* $R = \rho(l/A)$ and $X_L = 2\pi f N^2 \mu(A/l)$, we now include (where κ is the dielectric constant):

$$X_C = \frac{l}{2\pi f \kappa A} \qquad [17\cdot30]$$

17·12. Magnification of Harmonics. When the apparently sinusoidal emf of a commercial power supply is applied to a capacitor, an oscillogram of the current is likely to startle one who may not have previously considered the possibilities with some care. Such an oscillogram is given in Fig. 17·11.

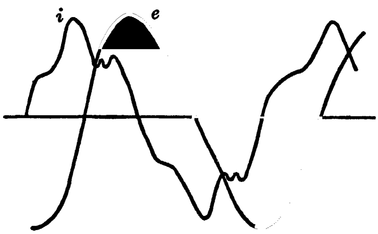


FIG. 17·11.

This is simply an example of the frequency discrimination of capacitive reactance as given by equation 17·29. The higher the frequency of any given harmonic component of applied voltage, the higher (per volt of the harmonic) will be the current of that frequency through the capacitor. The current wave therefore consists of the fundamental sine wave plus the harmonic components each amplified in proportion to the order (second, third, etc.) of the harmonic. In Fig. 17·11 there is five

times as much fifth harmonic in the current as there is in the voltage so that, while clearly present in the current, it is evident in the voltage only to the experienced observer.

With this example it becomes more sharply apparent why, as observed for item 1 of the preceding article, it is dangerous to deal with an *effective reactance* for nearly sinusoidal voltage and current.

In radio circuits it is often necessary to segregate the higher-frequency components of a composite current from those of lower frequency or from direct current. The frequency discrimination of capacitors is well adapted to this purpose, and so-called *by-pass condensers* are numerous in all kinds of radio and communication equipment. Connected across a resistor, the capacitor can provide a better path than the resistor for the high-frequency currents and by-pass them around the resistor. Care must be exercised to insure that both *blocking* and *by-pass* condensers have *sufficient* capacitance to avoid unduly high reactance for the *lowest* frequency which they are to pass. This is sometimes economically difficult.

17·13. Power and Energy for Reactance X_C. Substituting in $p = ei$ the particular values for capacitive reactance just considered we find

$$\begin{aligned} p_C &= e_C i_C = (E_m \sin \omega t)(I_m \cos \omega t) = E_m I_m \sin \omega t \cos \omega t \\ &= \tfrac{1}{2} E_m I_m \sin 2\omega t \end{aligned} \qquad [17·31]$$

This may be written

$$\begin{aligned} p_C &= \tfrac{1}{2}(X_C I_m) I_m \sin 2\omega t \\ &= \tfrac{1}{2} X_C I_m^2 \sin 2\omega t \end{aligned} \qquad [17·32]$$

which appears to be identical with p_L found for the inductive case in Chapter XV. Actually this is not quite true because we have taken $t = 0$ when $e = 0$ for the capacitive case, but $t = 0$ when $i = 0$ for the inductive case. The difference is seen by comparing Fig. 17·12 here with Fig. 15·17 of Chapter XV.

Like inductive reactance, however, the average power is

$$P_{\text{avg}} = 0 \qquad [17·33]$$

and the *maximum* power, known as the ***reactive power***, is

$$P_C = (P_C)_{\max} = \tfrac{1}{2} E_m I_m = \tfrac{1}{2} X_C I_m^2 \qquad [17·34]$$

The area under the *p-t* curve measures the energy stored in the capacitor. The maximum energy is most readily computed by

$$(W_C)_{\max} = \tfrac{1}{2} C E_m^2 \qquad [17·35]$$

and is represented by the area under one loop of the p-t curve. Again as for inductance, this energy is independent of frequency but, being stored and returned twice during each cycle, the *traffic* in energy (the power) is directly proportional to frequency.

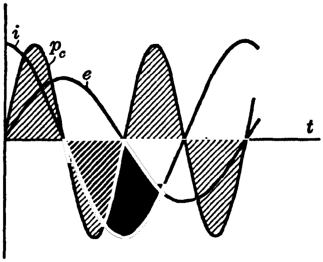


FIG. 17·12.

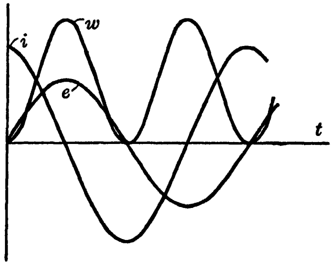


FIG. 17·13.

A curve of $w = \frac{1}{2}Ce^2$ is shown in Fig. 17·13. The average value of this curve is not zero, of course, and represents the amount of energy which, on the average, is "in hock" or hoarded by the capacitor. Depending upon circumstances, this hoarding of energy may either constitute an annoyance or serve a useful purpose, as will become apparent in later studies.

17·14. Resistance and Capacitance in Series for Alternating Current. When the current $i = f(t)$ through a series combination of resistance

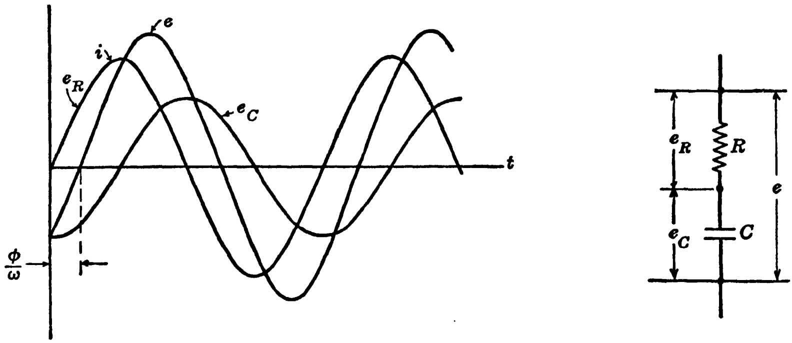


FIG. 17·14.

and capacitance is alternating, the voltage across the combination of course is the sum of the instantaneous voltages which have already been found for each component:

$$e = e_R + e_C \qquad [17\cdot36]$$

When the voltage $e = f(t)$ across the combination is given, the inverse problem of finding the current is, in general, too difficult for the

scope of this work. For the special case of sine waves, however, the solution is readily found by the same procedure as for the series RL circuit, because it is again a unique condition where the voltage and current have the *same* wave shape. The relations are most readily developed by beginning with the sinusoidal current.

17·15. For Sinusoidal Current. Given

$$i = I_m \sin \omega t \qquad [17\cdot37]$$

we have found

$$e_R = RI_m \sin \omega t = E_{mR} \sin \omega t \qquad [17\cdot38]$$

and

$$e_C = \frac{1}{C} I_m \int \sin \omega t \, dt = \frac{-1}{\omega C} I_m \cos \omega t$$

$$= -X_C I_m \cos \omega t = -E_{mC} \cos \omega t \qquad [17\cdot39]$$

Equation 17·36 gives

$$e = RI_m \sin \omega t - X_C I_m \cos \omega t \qquad [17\cdot40]$$

or

$$e = E_{mR} \sin \omega t - E_{mC} \cos \omega t \qquad [17\cdot41]$$

The same trigonometric process used in Chapter XV for the RL circuit is applicable here and gives

$$e = E_m \sin (\omega t + \phi) \qquad [17\cdot42]$$

Where E_m is an emf amplitude expressed by

$$E_m = \sqrt{E_{mR}^2 + E_{mC}^2} \qquad [17\cdot43]$$

or

$$E_m = I_m \sqrt{R^2 + X_C^2} \qquad [17\cdot44]$$

And phase angle ϕ is determined by

$$\phi = \tan^{-1}\left(-\frac{E_{mC}}{E_{mR}}\right) = \tan^{-1}\left(-\frac{X_C}{R}\right) \qquad [17\cdot45]$$

or

$$\phi = -\tan^{-1}\left(\frac{X_C}{R}\right) \qquad [17\cdot46]$$

The following statements are established.

1. The total voltage is sinusoidal.
2. The amplitude of this voltage is $E_m = \sqrt{(E_{mR})^2 + (E_{mC})^2} = I_m \sqrt{R^2 + X_C^2}$
3. This voltage *lags* the current by the phase angle $\phi = \tan^{-1}(+X_C/R)$. [This is the same as leading by angle $\phi = -\tan^{-1}(X_C/R)$.]

17·16. Impedance. We have seen that impedance for *sinusoidal quantities* is $Z = E_m/I_m$ where E_m and I_m are respectively the voltage and current amplitudes across and through some sourceless or *passive* portion of a circuit.

For this special case, Z is readily expressed in terms of R and X_C as follows (from equation 17.44):

$$Z = \frac{E_m}{I_m} = \sqrt{R^2 + X_C^2} \qquad [17\cdot47]$$

In the same way as for RL we find further that

$$E_m \cos\phi = I_m Z \cos\phi = RI_m, \quad \cos\phi = \frac{R}{Z} \qquad [17\cdot48]$$

$$E_m \sin\phi = I_m Z \sin\phi = -X_C I_m, \quad \sin\phi = \frac{-X_C}{Z} \qquad [17\cdot49]$$

By checking the earlier finding of statement (3) we note here,

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-X_C}{Z}\frac{Z}{R} = -\frac{X_C}{R}$$

Again we have material suggesting the use of a right triangle, called an *impedance triangle* as shown in Fig. 17·15.

As noted for RL, the triangle is *not* a *general* method of portraying impedance and it is *not* proper to define impedance as being $Z = \sqrt{R^2 + X_C^2}$; the general definition of impedance must be deferred until other matters have been studied.

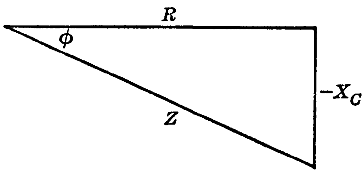


Fig. 17·15.

17·17. Power and Energy. To find the power we again substitute in $p = ei$ the particular functions of e and i for this particular case from equations 17·37 and 17·42:

$$p = ei = E_m \sin(\omega t + \phi) I_m \sin\omega t$$

or

$$p = E_m I_m \sin\omega t \sin(\omega t + \phi) \qquad [17\cdot50]$$

This has already been shown (Chapter XV) to give

$$p = E_m I_m[\tfrac{1}{2}\cos\phi - \tfrac{1}{2}\cos(2\omega t + \phi)]$$

$$p = \tfrac{1}{2}E_m I_m \cos\phi - \tfrac{1}{2}E_m I_m \cos(2\omega t + \phi) \qquad [17\cdot51]$$

A graph of $p = f(t)$ is shown in Fig. 17·16 and will be seen to differ from that for RL (see Fig. 15·23) because phase angle $\phi = -\tan^{-1}(X_C/R)$ is a minus angle or angle of *lag* instead of lead for voltage e.

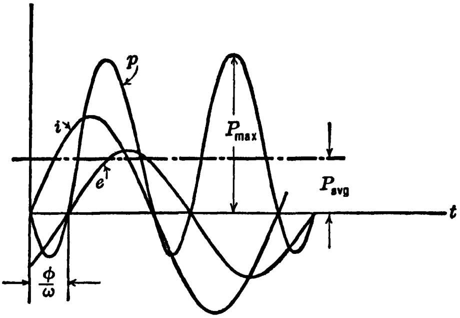


FIG. 17·16.

The value of maximum power $P_{\max}$ is easily found to be

$$P_{\max} = \tfrac{1}{2}E_mI_m[\cos\phi - (-1)] = \tfrac{1}{2}E_mI_m(1 + \cos\phi) \qquad [17\cdot52]$$

Attention is again called to the constant component of equation 17·51 because this is also the value of the ***average power.***

$$P_{\text{avg}} = \tfrac{1}{2}E_mI_m\cos\phi \qquad [17\cdot53]$$

Here again it is true that P_{avg} may be computed from $P_{\text{avg}} = \dfrac{1}{T}\displaystyle\int_0^T p\,dt$, but we utilize the labor-saving observation that ***the average value of any periodic quantity which varies symmetrically about a constant value is that constant value.***

Now, as usual, let us check this average *total* power to insure that it is entirely accounted for by the resistor heating as required for circuits with no other energy converting elements. This is accomplished by substituting equations 17·47 and 17·48 in 17·53 as follows.

$$P_{\text{avg}} = \tfrac{1}{2}(ZI_m)I_m\frac{R}{Z} = \tfrac{1}{2}RI_m^2 = P_R \qquad [17\cdot54]$$

17·18. Apparent, Real, and Reactive Power. Let us return to equation 17·44 and multiply it by $\frac{1}{2}I_m$ to obtain

$$\tfrac{1}{2}E_mI_m = \tfrac{1}{2}I_m^2\sqrt{R^2 + X_C^2}$$

or

$$(\tfrac{1}{2}E_mI_m)^2 = (\tfrac{1}{2}I_m^2R)^2 + (\tfrac{1}{2}I_m^2X_C)^2 \qquad [17\cdot55]$$

We have already observed that

$$P_R = \tfrac{1}{2}I_m^2 R \qquad [15\cdot 50]$$

and

$$P_C = \tfrac{1}{2}I_m^2 X_C \qquad [17\cdot 34]$$

Substituting these we find that

$$(\tfrac{1}{2}E_m I_m)^2 = P_R^2 + P_C^2 \qquad [17\cdot 56]$$

The first term $(\frac{1}{2}E_m I_m)$ is known as the ***apparent power*** or simply as the *volt-amperes* or *kva* (kilovolt-amperes). Expressed in words, equation 17·56 indicates that ***the square of the apparent power is equal to the sum of the squares of the average resistance power and the maximum reactance power.***

Recalling that we have designated the *maximum* reactive power to be ***the reactive power,*** and that the *average* resistance power is the only "working" power or ***real power,*** as it is usually called, we establish the important relation:

$$(\text{Apparent power})^2 = (\text{Real power})^2 + (\text{Reactive power})^2$$

While it may well seem to be a singular coincidence that this mixture of *average* and *maximum* power values should be so simply related it is well recognized and will be found to occupy an important place in engineering practice as well as forthcoming academic study.

17·19. Transfer of Energy from *C* to *R*. The discussion in Chapter XV of transfer of stored energy from reactance to resistance applies equally well to either inductive or capacitive reactance. Figure 15·25 should here be reviewed and the corresponding figure for RC constructed. Observe carefully the difference in phase relations, i.e., the timing of this energy transfer with reference to the current, for the two cases.

REFERENCES

1. KARAPETOFF, "The Electric Circuit."
2. *Westinghouse Engineer*, Vol. 1, pp. 8–11, May, 1941.

QUESTIONS

17·1. When the capacitor in Fig. 17·1 is being charged show what per cent of final value is attained by the capacitor voltage during the time $t = T$ where T = time constant.

17·2. Explain why the demonstration arrangement considered in Problem 15·2 would not work with capacitance in place of the inductance, and show what alteration will enable it to work.

17·3. Sketch a reasonably accurate oscillogram showing i, e_R, e_C for a series circuit of R and C which is periodically switched from a source of d-c potential E to a discharge resistor R_d and back again.

17·4. If the arrangement used in question 3 is operated with L in place of C, a high voltage is obtained for e_L when R_d is omitted ($R_d = \infty$). Explain fully what happens when $R_d = \infty$ as in the capacitive case of question 2.

17·5. Explain how the arrangement of question 3 can be operated to produce an approximate sawtooth oscillogram. Sketch and label the wave clearly.

17·6. Sketch an arrangement of hydraulic apparatus which is analogous to a capacitor with leakage. Expound the operation to simulate periodic charge and discharge from a source of constant potential.

17·7. Given an initially uncharged capacitor as in Fig. 17·1, find the time $t = f(R,C)$ from switch closure until maximum power is supplied to the capacitor.

17·8. Show that the value of the maximum power supplied to the capacitor in question 7 is independent of the value of the capacitance.

17·9. Show that the value of the maximum power supplied to the resistor in question 7 is independent of the value of the capacitance. When does this maximum occur?

17·10. How is a capacitor used in spot welding and what advantages does it have?

17·11. Make a graph like that for e in Fig. 17·7. If this represents the *current* instead of e for a capacitor determine the graph of e.

17·12. Given a sine current through a capacitance, determine by mathematical derivation the *voltage* (*a*) wave shape, (*b*) phase, (*c*) magnitude (as related to the *current* magnitude).

17·13. If the constant of integration in question 12 is not assumed zero, explain with a sketch of e and i versus t what the constant means. Give a practical example of a circuit for this case.

17·14. Give and define the unit of measure of capacitive reactance.

17·15. Explain why it is incorrect to state that alternating current through a capacitor leads the voltage across the capacitor by 90 degrees.

17·16. Sketch curves of R, X_L, X_C respectively versus frequency.

17·17. With a given capacitance and given maximum value of applied sinusoidal voltage, explain why the maximum power supplied to the capacitor increases with frequency although the maximum energy stored in the capacitor remains unchanged.

17·18. Explain why a capacitor, with a non-sine voltage applied, carries a current with relatively greater harmonic components than existed in the applied voltage.

17·19. Define *reactive power* and make clear its physical significance.

17·20. Show by circuit sketch how a *by-pass condenser* would be connected and explain its function.

17·21. Derive a relation which shows for a series connection of R and C carrying sinusoidal current:

(*a*) The phase angle between e and i: $\phi = f(R,C,f)$.

(*b*) The maximum emf $E_m = f(I_m,R,C,f)$.

17·22. In deriving equation 17·39, page 326, the constant of integration is tacitly assumed zero. With the aid of Fig. 17·14 and the physical aspects of the phenomenon, explain why this is or is not justified.

17·23. Given a series connection of R and C supplied from the same sinusoidal emf e as a series connection of R and L, sketch e, i_C, i_L, p_1, and p_2, where i_C = current through the capacitance, i_L = current through the inductance, p_1 = power supplied the RC circuit and p_2 = power supplied the RL circuit. Sketch the total power $p = p_1 + p_2$ and discuss the physical significance of the resulting curve of

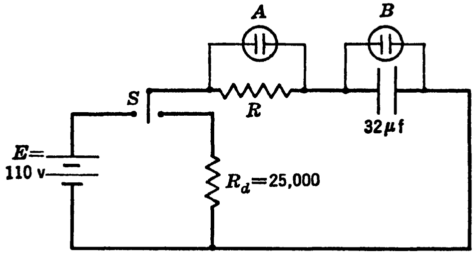
$p = f(t)$ from the point of view of the power company as discussed for inductive reactance on pages 272–273.

17·24. Derive for the capacitive case equations corresponding to 15·65 and 15·66.

PROBLEMS

17·1. The accompanying circuit uses neon lamps for a lecture demonstration of the behavior of the simple RC series circuit during charge and discharge. Neon lamps A and B each have:

Ignition voltage = 70 volts
Extinction voltage = 60 volts
Resistance approximately infinite.



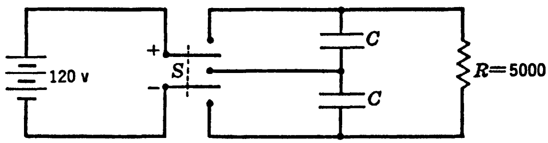
(*a*) Compute the time constant which will make lamp B light 3 sec after voltage E is applied.

(*b*) Compute the time at which lamp A will extinguish after E is applied.

(*c*) After the capacitor is fully charged switch S is moved to discharge position. What voltage appears across R_d at the instant discharge begins?

(*d*) How much energy is delivered to R_d?

17·2. By vibrating switch S to alternate equally between upper and lower positions, a voltage across R is produced which approaches twice the value of the 120 volts source as a limit. Battery circuit resistance and time for moving the switch may be neglected.

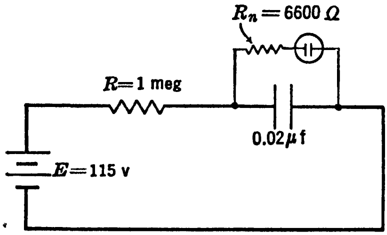


(*a*) Compute the value of C required to keep the voltage across R from falling below 220 volts when the switch is operated at 60 cycles per second.

(*b*) Explain why the practical functioning of this apparatus would actually *require* appreciable resistance in the battery circuit for reason entirely apart from the possibility that it might be impractical to make this resistance negligibly small.

17·3. The familiar neon glow lamp has an extinction potential V_e lower than V_i the ignition potential. For one type of commercial lamp rated at 2 watts on 115 volts, these are respectively 60 volts and 70 volts. This property enables the lamp to be used in the accompanying circuit with selected values of RC and E which cause it repeatedly to light and to extinguish when R exceeds the *critical value.* This value is where R will just pass sufficient current to maintain the lamp lighted at electrode potential V_e.

To obtain an approximate value for the frequency, the voltage between lamp electrodes may be neglected in favor of an equivalent resistance computed as follows:



$$R_n = V_e^2/P = (115)^2/2 = 6600 \text{ ohms.}$$

While the capacitor is charging to voltage V_i there is no lamp current. While the lamp glows from V_i to V_e the current through the 1-megohm resistance R is negligible in comparison with the lamp current so that the latter may be taken equal to the capacitor discharge current.

(*a*) Compute time during which the lamp is dark.

(*b*) Compute time during which the lamp is lighted.

(*c*) Compute frequency of flashing.

(*d*) How will increase of R affect the frequency?

(*e*) How will increase of C affect the frequency?

(*f*) How will increase of E affect the frequency?

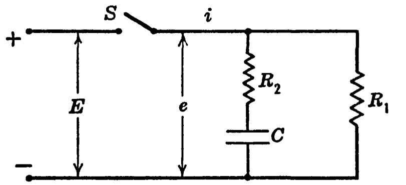
17·4. For the circuit illustrated below, compute:

(*a*) The time constant for charge (S closed), i.e., time that would be required for complete charge of capacitor if continued at the initial rate.

(*b*) The time constant for self-discharge (S open).

(*c*) The time required for charging current i to reach one-half the initial value.

(*d*) The time required for voltage e on discharge (S open) to reach one-half the initial value.



E = 500 volts
C = 5 μf
R_2 = 10 ohms
R_1 = 1 megohm

17·5. A series circuit of R = 1000 ohms and C = 2 μf (uncharged) is connected to a constant source of 100 volts.

(*a*) How long will it take, after application of the voltage, for the capacitor to acquire 38.6 per cent of the energy which it will have when fully charged?

(*b*) How much energy will the resistance have converted to heat by the time the capacitor is fully charged?

17·6. A capacitor of 159.2 μf has a sinusoidal voltage of 141.4 volts maximum value impressed across its terminals from a 50-cycle supply. Draw the waves of voltage and current and determine:

(*a*) The maximum value of the current.

(*b*) Maximum displacement Q "through" the capacitor for each half cycle.

(*c*) Maximum energy stored in the capacitor. At what points in the cycle does this occur?

(*d*) Maximum time rate at which the energy is stored in the capacitor.

17·7. A capacitor of 100 μf is connected across a sinusoidal a-c supply of 169 volts maximum value.

(*a*) What is the maximum value of the current that will flow at 25 cycles and at 50 cycles?

(*b*) What is the average power taken from the line in each case during one alternation of voltage?

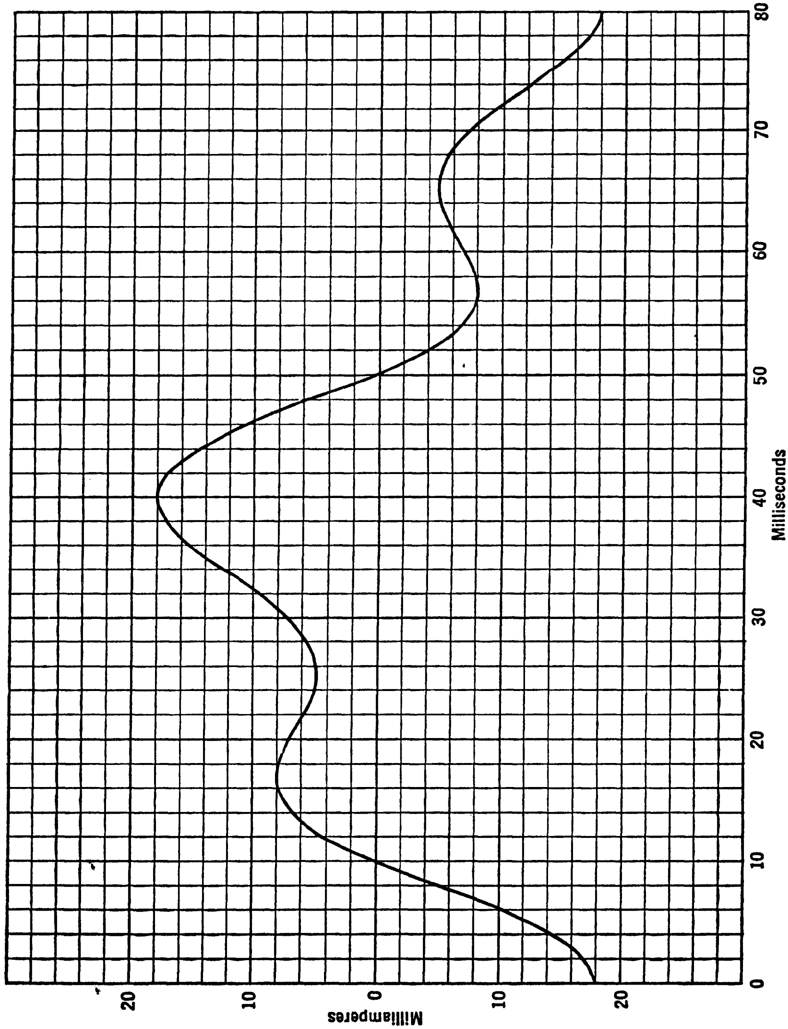
(*c*) What is the average power taken from the line in each case during one alternation of the *power?*

(*d*) What is the maximum displacement Q "through" the capacitor during each half cycle?

(*e*) What is the maximum energy stored in the capacitor?

17·8. Given the attached oscillogram of an alternating current through a pure capacitance of 4 μf. Determine graphically:

(*a*) The emf $e = f(t)$ across the capacitance for at least 16 instants of time and plot the e-t curve. Let $e = 0$ when $t = 0$, and one division on vertical axis equal 5 volts.



(b) The power $p_c = f(t)$ supplied to the capacitor. (1 division = 50 milliwatts.)

(c) The maximum displacement Q "through" the capacitor in each half cycle.

(d) The maximum energy stored in the capacitor. At what point (time) in the cycle does this occur?

(e) The maximum rate at which the energy is stored in the capacitor.

(f) The amount of the d-c component of the voltage.

17·9. The maximum power supplied from a sinusoidal source of emf to a circuit of R and C in series is 360 watts. The maximum energy stored in the capacitor is 50 millijoules. The heat power produced in the resistance is 90 watts. The impedance of the circuit is 30 ohms. Find:

(a) Phase angle ϕ between e and i.

(b) Resistance.

(c) Reactance.

(d) Maximum current.

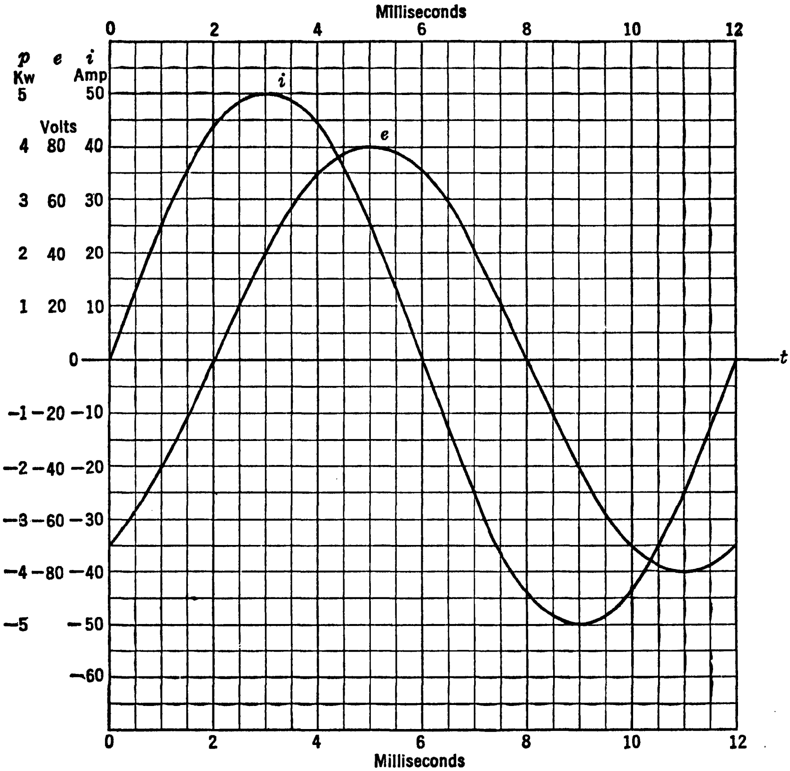
(e) Maximum emf across the circuit.

(f) Maximum emf across the capacitance.

(g) Capacitance.

(h) Frequency.

17·10. Given the accompanying sine waves of voltage and current for a series connection of R and C.



(a) By graphical process analyze e into the component emf's e_R and e_C across the resistance and capacitance respectively.

(b) By analytical process find $e_R = f(t)$ and $e_C = f(t)$.

(c) Compute R, X_C, C, and Z; check $\phi = \tan^{-1}(-X_C/R)$ with the graph.

(*d*) Obtain graphically the curves of total power p and the components p_R and p_C.

(*e*) Find the equation for $p = f(t)$ and compute the average power P_{avg}.

(*f*) Compute the energy in joules converted into heat during one cycle.

(*g*) Compute the total (not the net) energy *supplied* to impedance Z from the power supply during one cycle, i.e.., the above-axis area under p in (*d*).

(*h*) Compute the total of energy actually *returned* from impedance Z to the power supply during one cycle and check against the data from (*f*) and (*g*).

(*i*) Compute the ratio of energy in (*f*) to energy in (*g*), that is, the fraction of the gross energy supplied which is useful (as heat).

(*j*) Compute the ratio of energy in (*f*) to the total energy traffic ($g + h$), that is, the fraction of the total energy traffic which is useful (as heat).

(*k*) Compute the energy transferred from C to R during one cycle.

(*l*) Compute from (*k*) the per cent of energy stored in C which is transferred to R.

Chapter XVIII

SERIES CIRCUITS WITH *R*, *L*, AND *C*

18·1. The Three Parameters. The three quantities which have now been introduced, resistance, inductance, and capacitance, constitute the basic parameters of all kinds of electric circuits and of many kinds of electric equipment; they characterize the physical constitution of the circuit and predestine the events which will result from an electrical disturbance if and when the disturbance occurs. They have two aspects, constructional and functional, as follows in summary:

Parameter	Constructional	Functional	
Resistance	$R = \rho \dfrac{l}{A}$	$R = \dfrac{e}{i}$	$W = Ri^2 dt$
Inductance	$L = N^2 \mathcal{P} = N^2 \mu_s \mu_0 \dfrac{A}{l}$	$L = \dfrac{N\phi}{i} = \dfrac{e}{di/dt}$	$W = \frac{1}{2}Li^2$
Capacitance	$C = \kappa_s \kappa_0 \dfrac{A}{l}$	$C = \dfrac{q}{e} = \dfrac{i}{de/dt}$	$W = \frac{1}{2}Ce^2$

The "Constructional" indicates the physical constitution, that is, the materials and the dimensions to which they are fashioned.

The "Functional" indicates the predestined relation among the participating electrical forces of nature, that is, the relation between *cause* and *effect*.

18·2. The Series Circuit. Any purely *series* combination of resistances, inductances, and capacitances can be reduced to an equivalent series connection comprising one each of R, L, and C as in Fig. 18·1. The total voltage e at any instant is, by Kirchhoff's law,

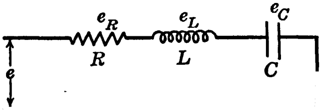


FIG. 18·1. A series *RLC* circuit.

$$e = e_R + e_L + e_C \tag{18·1}$$

$$e = Ri + L\frac{di}{dt} + \frac{1}{C}q \tag{18·2}$$

$$e = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q \qquad [18\cdot3]$$

or

$$\frac{de}{dt} = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i \qquad [18\cdot4]$$

This equation is considerably more formidable for solution than was that for either R and L, or R and C. Physically, this is the consequence of combining in one circuit two kinds of energy storage. *Kinetic* energy is stored in the inductance *during* the flow of current; *potential* energy is stored in the capacitance as a *consequence* of integrated current flow. This makes possible an oscillatory type of current flow which alternately transfers the energy to and from one or the other type of storage in much the same manner as occurs for the mechanical energy in a pendulum, in the reeds of certain musical instruments, and in other examples of mechanical oscillation.

Given a curve $i = f(t)$, it is but an extension of the graphical processes previously discussed to find the total voltage $e = e_R + e_L + e_C$. Unfortunately this is not the practical situation commonly encountered; it is usual to have given $e = f(t)$, either (oscillo)graphically or analytically, to find $i = f(t)$. Except for a few special but very prevalent cases, this is an extremely difficult computation. The simplest case is encountered with sinusoidal emf or current as will now be developed.

18·3. Sinusoidal Current. First let $i = I_m \sin \omega t$. From previous study we know that

$$e_R = RI_m \sin \omega t$$

$$e_L = X_L I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

and

$$e_C = X_C I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

It is a simple process, then, to find the total voltage.

$$e = e_R + e_L + e_C = RI_m \sin \omega t + X_L I_m \sin\left(\omega t + \frac{\pi}{2}\right) + X_C I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= I_m(R \sin \omega t + X_L \cos \omega t - X_C \cos \omega t)$$

$$e = I_m[R \sin \omega t + (X_L - X_C) \cos \omega t] \qquad [18\cdot5]$$

The form of this equation has already been encountered more than once. Its solution is now well-known to be

$$e = E_m \sin(\omega t + \phi) \qquad [18\cdot6]$$

where

$$E_m = I_m\sqrt{R^2 + (X_L - X_C)^2} \qquad [18\cdot7]$$

and

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} \qquad [18\cdot8]$$

The impedance, of course, is

$$Z = \frac{E_m}{I_m} = \sqrt{R^2 + (X_L - X_C)^2} \qquad [18\cdot9]$$

Note carefully that there are only *two* squared terms

$$Z \neq \sqrt{R^2 + X_L^2 + (-X_C)^2} \qquad [18\cdot10]$$

From these relations it is fairly apparent that the effects of inductive reactance X_L and capacitive reactance X_C are "antagonistic" and that the one is effective directly in nullifying an equal amount of the other insofar as the *overall* performance is concerned. Natural curiosity prompts us to inquire what happens when $X_L = X_C$.

18·4. When $X_L = X_C$. The consequence of making $X_L = X_C$ is best observed first in the impedance. Substituting $X_L - X_C = 0$ in equation 18·9 gives

$$Z = \sqrt{R^2 + 0} = +R \qquad [18\cdot11]$$

(only $+R$ is admissible because $-R$ has no physical meaning here). This is to say that the circuit as a whole behaves as though the reactances were not present; that the total voltage and the voltage across the resistor are identical, $e = e_R$. If a sinusoidal voltage e is impressed across the whole circuit, the same current will flow as for the resistance alone, $i = e/R$.

Although it is now clear that $e_L + e_C = 0$, it does *not* follow that these voltages are *each* zero and that there will be no voltage across either reactance X_L or X_C. Both e_L and e_C depend upon the current as well as upon L and C respectively, and when R is sufficiently small the current may produce e_L and e_C even greater than the total voltage e so that for maximum values we have $E_{mL} = E_{mC} > E_m = E_{mR}$. This is shown in Fig. 18·2.

When the resistance R is reduced, the current I_m for a given voltage E_m will increase proportionally and the reactance voltages ($E_{mL} = E_{mC} = XI_m$) will increase likewise. It is entirely possible for these reactance

voltages to become so much larger than the applied voltage E_m as to constitute a *real hazard* to life and equipment; capacitors may well break down and even likewise the insulation of inductance coils.

On the other hand this magnification or *amplification* of voltage may be put to good use, as is done in the familiar radio receivers which receive voltages so weak as to be quite innocent of physical hazard even when amplified to a far higher degree than can be attained with the most favorable practical magnitudes of R, X_L, and X_C now obtainable.

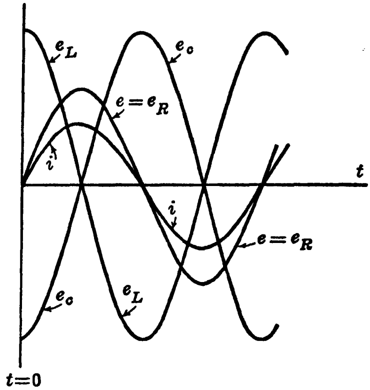


FIG. 18·2.

18·5. Resonance. The very special case just considered is *one* of resonance. In general, ***a resonant circuit is one where the current and impressed voltage will be in phase for the specified frequency, yet not so for other frequencies.***

The role of frequency in the series circuit is well portrayed in Fig. 18·3 where Z versus f is developed from equation 18·9 which may be expanded into

$$Z = \sqrt{R^2 + \left(2\pi Lf - \frac{1}{2\pi Cf}\right)^2} \qquad [18 \cdot 12]$$

It is a simple matter to find the frequency for resonance (f_r), dotted in Fig. 18·3, by solving the expanded $X_L = X_C$ as follows.

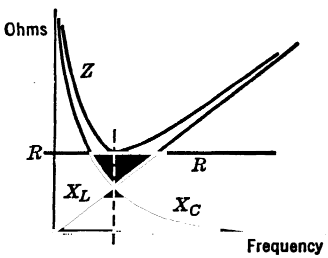


FIG. 18·3.

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$f_r = \frac{1}{2\pi\sqrt{LC}} \qquad [18 \cdot 13]$$

Because both equations 18·11 and 18·13 were obtained for $X_L = X_C$, and because it is evident that equation 18·11 (for constant R) gives the minimum value of $Z = f(f)$, it is established that, for this particular circuit, resonant frequency f_r is accompanied by minimum circuit impedance Z.

18·6. A Mechanical Analog. It is well to observe that a useful analogy exists between this case of electrical resonance and the mechanical resonance of a vibrating member such as a reed or cantilever spring. The

resonant frequency of the reed is expressed by the same equation 18·13, provided L and C are taken to mean respectively the effective *inertia* and *compliance* (reciprocal stiffness) of the reed. In accord with everyday experience the equation shows that either *more* inertia (L) or *more* compliance (C) (less stiffness) *reduces* the resonant frequency (f_r) of the mechanical vibration.

Mechanically, too, it is common experience that innocently small forces, periodically applied to a resonant structure at resonant frequency, may produce greatly magnified forces possibly sufficient to fracture the structure. This is analogous to the situation where electrical resistance is small enough to make $E_{mL} = E_{mC} > E_m = E_{mR}$ as discussed earlier.

18·7. Power and Energy. The earlier studies of power separately for R, L, and C apply directly to the series circuit of all three. We have found that when $i = I_m \sin \omega t$,

For R, $$p_R = \tfrac{1}{2}RI_m^2 - \tfrac{1}{2}RI_m^2 \cos 2\omega t \qquad [15\cdot 48]$$

For L, $$p_L = \tfrac{1}{2}X_L I_m^2 \sin 2\omega t \qquad [15\cdot 42]$$

For C, $$p_C = -\tfrac{1}{2}X_C I_m^2 \sin 2\omega t \qquad [17\cdot 32]$$

The total power is $p = p_R + p_L + p_C$. Since $X_L = X_C$ for resonance here, the sum $p_L + p_C$ is evidently zero because $p_L = -p_C$. This leaves $p = p_R$ as might be expected from previous study. In other words, there is only a *local* transfer of energy from inductance to capacitance and back again which does not involve the circuit as a whole and never becomes manifest outside of the circuit. The situation is illustrated by the graphs in Fig. 18·4.

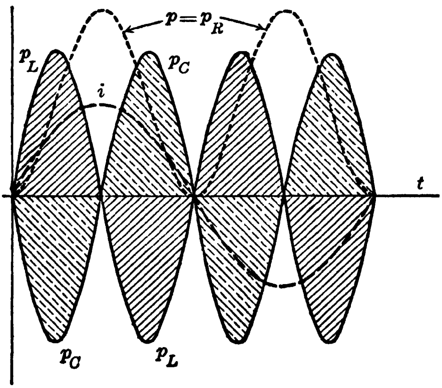


FIG. 18·4.

When the circuit is not resonant $p_L \neq p_C$ and $p \neq p_R$. Construction of the graph similar to Fig. 18·4 is left for the student.

18·8. Resonance in General. It would be unpardonably misleading to introduce the subject of electrical resonance by means of the simple series RLC case without observing that this is a very *special* case even though it is one which is very commonly encountered with acceptable approximation. Although it is beyond the scope of this work to deal with the general case, it will later be found that except for a very few special cases, neither minimum impedance nor $X_L = X_C$ occur at the resonant frequency which we have defined to cover the general case. Particular care then is required at this stage of our study to avoid a temptation to place $X_L = X_C$ on mental file as synonymous with resonance. Let us attach a large red label: ***"$X_L = X_C$ for resonance, special case only, handle with care!"***

18·9. D-C Applied to RLC in Series. Let us now consider the current and voltages produced in the series circuit of R, L, and C when a d-c voltage is applied in the same manner as was done with RL and with RC. If switch S in Fig. 18·5 is closed when $t = 0$, the initial current is $i = 0$.

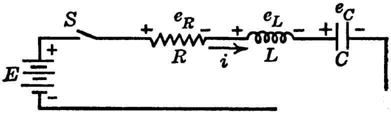


FIG. 18·5.

For simplicity let us consider the capacitor to be initially uncharged, so $q = 0$ when $t = 0$. The equation is

$$E = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q \qquad [18\cdot14]$$

Rearranging,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E \qquad [18\cdot15]$$

Differentiating,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0 \qquad [18\cdot16]$$

18·10. Solution of the Equation. The solution to this particular equation is well known and may be obtained by any of several methods common in the study of differential equations. Only the result will be given here:

Case I

$$i = \frac{E}{2AL}\epsilon^{-\frac{R}{2L}t}(\epsilon^{At} - \epsilon^{-At}) \qquad [18\cdot17]$$

where

$$A = \left[\frac{R^2}{4L^2} - \frac{1}{LC}\right]^{1/2} \qquad [18\cdot18]$$

So long as $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$, in equation 18·18 or

$$R^2 > 4\frac{L}{C} \qquad [18\cdot 19]$$

the quantity A is an ordinary number and the evaluation of $i = f(t)$ is easily obtained together with the curve of Fig. 18·6*a*.

The curve starts in the same manner as that for R and L without C (cf. Fig. 15·4). This is reasonable because, with $q = 0$ when $t = 0$, the capacitor voltage is zero and the capacitor offers no opposition to the initial current flow.

The initial slope of the current can readily be shown to be E/L which is identical with that for RL as discussed in connection with the time constant. As the capacitor voltage builds up from the current flow, less voltage is available across the RL elements. Consequently the current falls off from the RL exponential rise which it at first followed and reaches a maximum value short of the previously significant $I = E/R$ value. The capacitor, as it acquires a larger and larger share of the available voltage of the constant supply, acquires more and more control of the current so that the latter portion of the curve approaches the zero axis in the manner of the RC exponential without L (cf. Fig. 17·2*a*). In short we find that, for a given E, L controls the early period of current flow and gradually relinquishes control to C which controls the latter period of current flow.

Case II. When, in equation 18·18, $R^2/4L^2 < 1/LC$, or

$$R^2 < 4\frac{L}{C} \qquad [18\cdot 20]$$

the quantity A involves $\sqrt{-1}$ which is an imaginary instead of a real number. Fortunately it is possible to utilize certain mathematical relations which translate the seemingly unreal into a form which restores evidence of reality to it, as follows.

If we let

$$B = \left[\frac{1}{LC} - \frac{R^2}{4L^2}\right]^{1/2} = \left[-1\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)\right]^{1/2} = \sqrt{-1}\left[\frac{R^2}{4L^2} - \frac{1}{LC}\right]^{1/2} \qquad [18\cdot 21]$$

we have made $B = A\sqrt{-1}$ and B is a real number when $R^2 < 4(L/C)$.

Furthermore,

$$A = \frac{B}{\sqrt{-1}} = -B\sqrt{-1} = -jB$$

where $j = \sqrt{-1}$.

Equation 18·17 may now be written

$$i = \frac{E}{-2jBL}\,\epsilon^{-\frac{R}{2L}t}(\epsilon^{-jBt} - \epsilon^{jBt})$$

$$= \frac{E}{2BL}\,\epsilon^{-\frac{R}{2L}t}(\epsilon^{jBt} - \epsilon^{-jBt})\,\frac{1}{j} \qquad [18 \cdot 22]$$

Now it is a known mathematical fact that

$$(\epsilon^{jBt} - \epsilon^{-jBt})\,\frac{1}{j} = 2\sin Bt \qquad [18 \cdot 23]$$

Substituting this in equation 18·22 we find

$$i = \frac{E}{BL}\,\epsilon^{-\frac{R}{2L}t}\sin Bt \qquad [18 \cdot 24]$$

Note that the current is ***sinusoidal with fixed frequency,*** but with decreasing amplitude governed by $\epsilon^{-Rt/2L}$ (dotted) as shown in Fig. 18·6*b*.

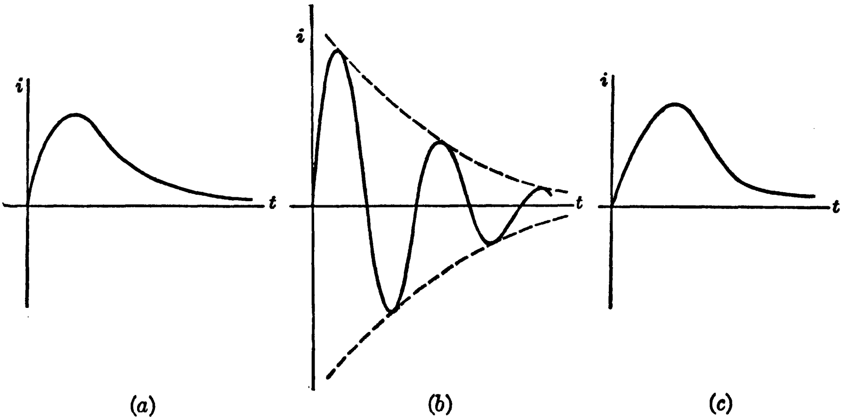


FIG. 18·6.

The physical circumstances which give rise to the oscillatory condition are most readily analyzed in terms of the *energies* involved. With reduced R, and a given ratio L/C, the current attains a higher maximum value so that the kinetic energy $W_L = \frac{1}{2}LI^2$ acquired by the inductance is greater. Before the current decreases to zero this energy must be

entirely converted, in part by i^2R heating and in part to capacitive storage ($\frac{1}{2}Ce^2$). Since the final capacitor voltage is necessarily that of the supply E, the amount of energy which can finally be stored in the capacitor is strictly limited to the value $W_C = \frac{1}{2}CE^2$ as determined only by capacitance C and by voltage E applied to the circuit. By sufficient reduction of R, the kinetic energy acquired by L will exceed that which can be taken care of by the reduced heating (i^2R) and by the limited capacitor storage. The current then reaches zero only by overcharging the capacitor with the excess energy ($\frac{1}{2}Li^2$) which the inductance has to get rid of before zero current is reached. The capacitor can unload this excess energy only by means of a reverse flow of current, which it has the necessary voltage to produce. Consequently the current decreases through zero and increases in reverse, as seen in Fig. 18·6*b*. Again the kinetic energy of the inductance becomes large because of high current flow in the low-resistance circuit, and depletes the capacitor energy below $W_C = \frac{1}{2}CE^2$ before current flow is brought to zero. With applied voltage now greater than the capacitor voltage, current again flows in the original direction only to repeat an overcharge of capacitor energy. Thus the cycle is repeated, always overshooting, but by an amount diminishing as the unwanted energy is gradually dissipated in i^2R heating until the oscillation may be considered to be damped out to a practical (if not mathematical) zero value.

The relation $R^2 < 4L/C$, which characterizes the oscillatory condition, indicates that oscillation can be prevented by a sufficient *increase* in either R or C, or by a sufficient *decrease* in L. This is confirmed by the physical analysis because:

1. Larger R will serve both to slow up the current, so that less kinetic energy $W_L = \frac{1}{2}LI^2$ will be acquired, and to dissipate the energy faster in i^2R heating, until reversal of current may be avoided.
2. Larger C will provide greater energy storage $W_C = \frac{1}{2}CE^2$ for the given E so that all the kinetic energy can be accommodated and there will be no excess to require reversal of current.
3. Smaller L will reduce the kinetic energy $W_L = \frac{1}{2}LI^2$ until the excess can be avoided as in (2).

Case III. When, in equation 18·18

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

or

$$R^2 = 4\frac{L}{C} \qquad [18\cdot 25]$$

both A (equation 18·18) and B (equation 18·21) are apparently zero and both equations 18·17 and 18·24, as they stand, are seemingly unable to provide a sensible answer. This is known as the *critical case* and makes necessary the aid of mathematical processes which need not be given here to evolve the form

$$i = \frac{E}{L} t \epsilon^{-\frac{R}{2L}t} \qquad [18\cdot 26]$$

The i-t graph (Fig. 18·6c) is nonoscillatory and comparable with that of Case I (Fig. 18·6a).

The three cases may now be tabulated as follows.

Case	Criterion	$i = f(t)$
I Nonoscillatory	$R^2 > 4\frac{L}{C}$	$i = \frac{E}{2AL} \epsilon^{-\frac{R}{2L}t} (\epsilon^{At} - \epsilon^{-At})$
II Oscillatory	$R^2 < 4\frac{L}{C}$	$i = \frac{E}{BL} \epsilon^{-\frac{R}{2L}t} \sin Bt$
III Critical	$R^2 = 4\frac{L}{C}$	$i = \frac{E}{L} t \epsilon^{-\frac{R}{2L}t}$

18·11. The Component Voltages. All the voltages $e_R = f(t)$, $e_L = f(t)$, and $e_C = f(t)$ are of interest. Knowing $i = f(t)$ now, all voltages are readily determined from the familiar basic e versus i relations for R, L, and C respectively.

18·12. Circuit Shorted, *E* Removed. If as in Fig. 18·7 the voltage E is removed and the circuit again closed by switch S, the energy stored in the capacitor, which was charged to voltage E, now for a time sustains current. Except for direction the current is exactly as found during charge, because the equation is the same as 18·16 except for the modified initial condition that $q \neq 0$ when $t = 0$ and $i = 0$. The table just given for $i = f(t)$ therefore applies to this circuit when the algebraically plus direction of current is taken as shown in Fig. 18·7.

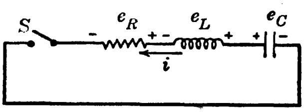
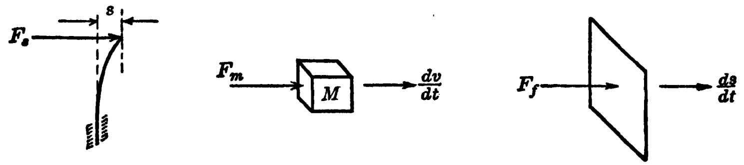


Fig. 18·7.

18·13. A Mechanical Analog for *RLC* in Series. It is commonly helpful, in sharpening the physical concept of the RLC circuit behavior just discussed, to study an analogous mechanical situation. Let us consider a vibrating reed as in Fig. 18·8a which requires force F_s to displace or spring it distance s according to $F_s = Ks$; this is analogous to capacitive phenomena. Let Fig. 18·8b represent mass M which requires force

F_M to accelerate it at rate $dv/dt = d^2s/dt^2$ according to $F_M = M(d^2s/dt^2)$; this is analogous to inductive phenomena. Let Fig. 18·8*c* represent dynamic friction which we will postulate to be according to the relation



(*a*) (*b*) (*c*)

Fig. 18·8.

$F_f = R(ds/dt)$ so that the analogy with electrical resistance will be evident.

When these components are combined as in Fig. 18·9, the total force is the sum of the component forces so that

$$F = F_f + F_M + F_s \tag{18·27}$$

$$F = R\frac{ds}{dt} + M\frac{d^2s}{dt^2} + Ks \tag{18·28}$$

$$\frac{dF}{dt} = R\frac{dv}{dt} + M\frac{d^2v}{dt^2} + Kv \tag{18·29}$$

The mechanical combination is analogous to a *series* electrical connection because the mechanical velocities, like the electric currents, are *identical* for all component parts of the combination or connection.

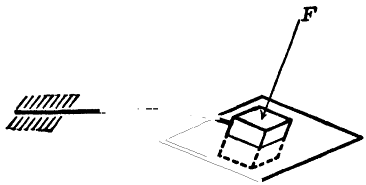


Fig. 18·9.

Now let a constant force F, such as the force of gravity on mass M, be applied as in Fig. 18·9, with displacement s initially zero. Because of inertia, the initial velocity $v = ds/dt$ must also be zero. Because of the spring action, the final displacement must be $S = F_s/K = F/K$. During the intervening time, however, either of two phenomena may occur, as follows.

1. If the friction is sufficient to keep the velocity below some critical value, the displacement will simply proceed with an acceleration followed by a deceleration as in Fig. 18·10*a* until the final position or displacement has been achieved. This corresponds to Fig. 18·6*a* or 18·6*c* for the electric current.

2. If friction is insufficient for (1) the deceleration will fail to stop the motion at $S = F/K$ and the displacement will overshoot so that $F_s > F$. The ensuing oscillation or vibration, as shown in Fig. 18·10*b*,

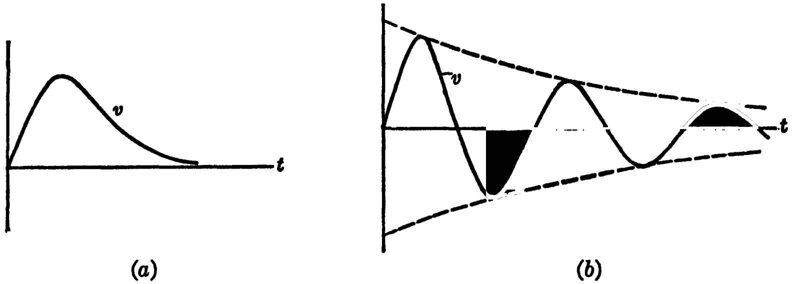


FIG. 18·10.

continues until the excess stored energy, which was acquired by the mass when accelerated to excessive velocity, can be reduced by friction to the amount storable in the spring with deflection $S = F/K$. For the electric current this, of course, corresponds to Fig. 18·6*b*.

REFERENCES

1. ESHBACH, "Handbook of Fundamentals," pp. 8-61–8-62.
2. "Standard Handbook for Electrical Engineers," McGraw-Hill Book Co., Seventh Edition, pp. 2-143–2-152.

QUESTIONS

18·1. Explain what is meant by the designation of electric circuit relations as *constructional* and *functional*.

18·2. Cite an example and show how a purely series combination of more than one each of R, L, and C is reducible to Fig. 18·1.

18·3. Define for a series circuit of R, L, and C the circumstances at resonance which produce, across one or more of the elements, a larger sinusoidal voltage than is applied to the circuit.

18·4. Analyze the conditions when the circuit of question 3 is not operated at resonant frequency and determine whether it is possible that (E_{mC} or E_{mL}) $> E_m$ for this case.

18·5. Sketch curves similar to those of Fig. 18·4 for the case where $X_L > X_C$.

18·6. Given equation 18·12 show, by computing $dZ/df = 0$, that the minimum impedance of the series circuit is $Z_{\min} = R$.

18·7. In Case III derive an expression for the maximum current and the time when it occurs, each in terms of E, R, L.

18·3. In Case II it is stated that "B is a real number when $R^2 < 4L/C$." In terms of R, L, C, what is this number?

18·9. Substitute the function $i = f(t)$ found for each case (I, II, III) into the parent equation (18·16) and show that it satisfies the equation.

18·10. For Case III compute $e_R = f(t)$, $e_L = f(t)$, $e_C = f(t)$ and sketch the curves along with $i = f(t)$ as given in Fig. 18·6c.

18·11. For one cycle of the current in Fig. 18·6b show for each instant of zero current and for each instant of maximum current the current *direction* and the *polarity* of each emf E, e_R, e_L, e_C. Use four sketches of the circuit, one for each successive instant, and clearly identify each.

PROBLEMS

18·1. Given for the circuit of Fig. 18·1:

$$i = 20 \sin (377t) \qquad L = 0.0848 \text{ henry}$$

$$R = 18 \text{ ohms} \qquad C = 166\ \mu\text{f}.$$

Determine:

(*a*) The numerical equation for the applied voltage e.
(*b*) The average power taken by the circuit. Compute this by two methods.
(*c*) The angle between the current and the applied voltage e.
(*d*) The total energy *traffic* between L and C during *each cycle* of impressed voltage.

18·2. For the circuit of Fig. 18·1 the following data are given.

For frequency $f = 1000$ cycles per second

$$i = 10 \sin \omega_1 t$$

$$e = 100 \sin \left(\omega_1 t + \frac{\pi}{6}\right)$$

For frequency $f = 500$ cycles per second

$$i = 8.2 \sin \omega_2 t$$

$$e = 100 \sin \left(\omega_2 t - \frac{\pi}{4}\right)$$

(*a*) Sketch curves of e and i versus t at 1000 cycles per second. Compute:
(*b*) R, L, and C.
(*c*) f_r (resonant frequency).
(*d*) P_{avg} (at 1000 cycles per second).

18·3. Given a d-c voltage of 110 volts applied to a series circuit of R, L, and C with values $C = 0.5\ \mu$f, $L = 0.18$ henry, and the value of resistance R required for the *critical* case. Let the circuit be initially dead. Following application of the given voltage at $t = 0$, compute:

(*a*) Time t at which I_{max} occurs.
(*b*) Maximum current flow.
(*c*) Maximum energy stored in the inductance.
(*d*) Maximum energy stored in the capacitor.
(*e*) The numerical equation for $e_R = f(t)$, voltage across the resistance.
(*f*) The numerical equation for $e_L = f(t)$, voltage across the inductance.

APPENDIX I

TABLE I. Selected Conversion Factors

Length

1 in. = 2.54 cm
1 mil = 0.001 in.

Mass

1 lb = 453.6 gm
1 kg = 2.2 lb

Force

1 newton = 0.2248 lb
= 10^5 dynes

Energy

1 joule = 1 watt-sec
= 1 coulomb-volt
= 1 newton-meter
1 kwh = 3413 Btu
= 1.341 hp-hr
1 ft-lb = 1.356 joules
1 Btu = 1055 joules
= 778 ft-lb
= 252 gm-cal
1 lb-cal = 1900 joules
1 kg-cal = 4180 joules

Volume

1 gal = 3785 cu cm
= 231 cu in.
= 3.785 liters
1 liter = 1.057 qt
= 61.02 cu in.

Pressure

1 atmosphere = 14.70 lb/sq in.
= 29.92 in. of mercury
= 33.90 ft of water

Power

1 watt = 1 joule/sec
= 44.26 ft-lb/min
= 0.01434 kg-cal/min
= 0.0569 Btu/min
1 HP = 550 ft-lb/sec
= 746 watts

Miscellaneous

1 cu ft of water = 62.4 lb
1 gal of water = 8.33 lb
$\log_{10} N = 2.303 \log_e N$
Area of circle $A = \pi r^2$
Surface of sphere $A = 4\pi r^2$
Volume of sphere $V = \frac{4}{3}\pi r^3$

TABLE II. SOME GENERAL PHYSICAL CONSTANTS

Velocity of light	$c = 2.99776 \times 10^8\ m/\text{sec}$
Velocity of sound in air	1,129 ft/sec at 20° C
Electron charge	$e = 16.020 \times 10^{-20}$ coulomb
Electron mass	$m = 9.11 \times 10^{-31}$ kg
Mass of H atom	1.67×10^{-27} kg
Ratio electron charge to mass	$e/m = 1.759 \times 10^{11}$ coulombs/kg
Faraday constant	$F = 96{,}487$ coulombs/chem. equiv.
Planck constant	$h = 6.62 \times 10^{-34}$ joule-sec
Avogadro's number	$N_0 = 6.023 \times 10^{23}$ molecules/mole
Molar gas constant	$R_0 = 8.315$ joules/mole °K
Boltzmann constant R_0/N_0	$k = 1.380 \times 10^{-23}$ joule/molecule °K
Acceleration of gravity (earth)	$g = 9.8\ m/\text{sec}^2 = 32.2\ \text{ft/sec}^2$
Base of natural logarithms	$\epsilon = 2.7183$
Ratio of natural to common log	$\dfrac{\log_e x}{\log_{10} x} = \dfrac{\ln x}{\log x} = 2.30259$
Ratio of circumference to diameter of circle	$\pi = 3.14159265$

TABLE III. PHYSICAL PROPERTIES OF METALS

Name	Atomic Weight	Density: G per cu cm	Density: Temp,* deg cent	Melting Point, deg cent	Latent Heat of Fusion, g-cal per g	Boiling Point, deg cent	Latent Heat of Vaporization, g-cal per g	Specific Heat: G-cal per g or Btu per lb	Specific Heat: Temp, deg cent	Thermal Coefficient of Linear Expansion: $\times 10^4$ per deg cent	Thermal Coefficient of Linear Expansion: Temp, deg cent	Thermal Conductivity: G-cal per sec per sq cm per deg cent per cm	Thermal Conductivity: Temp, deg cent	Electrical Resistivity: Microhm-cm	Electrical Resistivity: Temp,* deg cent	Temp Coefficient of Resistivity: Temp Coefficient	Temp Coefficient of Resistivity: Temp, deg cent
Aluminum....	26.97	2.70	20	659.8	93.0	1800	1950–2000	0.226	0–100	0.257	20–300	0.480	18	2.688	20	0.00403	20
Antimony....	121.76	6.618	20	630.5	39.0	1380	373	0.0504	20–100	‖0.136 ⊥0.080	20 20	0.0442	0	39.1	0	0.0036	20
Arsenic.......	74.93	5.73	14	Volatilizes	....	615 Sublimes	74	0.078	18	0.05	20	....	......	35	0	0.0042	20
Barium.......	137.36	3.5	20	850	....	1140	628	0.068	−185 to +20	....	......	....	......	9.8	20	0.0033	20
Beryllium....	9.02	1.8	20	1350	318	1500	....	0.425	0–100	0.122	20	....	......	10.1	20	....	....
Bismuth......	209.00	9.781	20	271.3	12.5	1450	221	0.0294	20	‖0.140 ⊥0.103	20 20	0.0194	18	119.0	18	0.004	20
Cadmium.....	112.41	8.648	20	320.9	12.8	766	227.5	0.0552	27.9	‖0.54 ⊥0.20	0 0	0.222	18	7.54	18	0.0042	0
Calcium......	40.08	1.55	20	810	78	1170	....	0.149	0–100	0.25	0–21	....	......	4.59	20	0.00364	0–600
Cerium.......	140.13	6.90	20	640	....	1400	....	0.0511	20–100	....	......	....	......	78	20	....	....
Cesium.......	132.81	1.873	20	26.0	3.78	670	131.4	0.0482	0–26	0.97	0–26	....	......	19	0	0.00478	−80 to +25
Chromium....	52.01	6.93	25	1765	70	2200	14.71	0.111	18–100	0.068	20–100	....	......	2.6	0	....	....
Cobalt.......	58.94	8.71	21	1480	64	3000	....	0.1001	20	0.123	20	....	......	9.7	20	0.00658	0–100
Copper.......	63.57	8.89	20	1083	49.3	2300	1756	0.0928	18–100	0.162	20	0.918	18	1.724	20	0.00393	20
Gold.........	197.2	19.3	20	1063	15.9	2600	446	0.0312	18	0.143	17–100	0.705	17	2.44	20	0.0034	20
Iridium......	193.1	22.42	17	2454±3	26.1	4800	340	0.0323	18–100	0.065	20	0.141	17	6.10	0	0.00411	0–100
Iron (99.97%)	55.84	7.87	20	1535	65	3200	1110	0.1075	20	0.119	0–100	0.18	0	9.8	20	0.0065	0–100
Lead.........	207.22	11.342	20	327.4	6	1620	323	0.0297	0	0.291	20–100	0.083	18	22.0	20	0.0039	20
Lithium......	6.940	0.534	20	186	120.5	1200	....	0.96	50	0.512	0–178	0.167	0	8.55	0	0.0047	0
Magnesium...	24.32	1.74	20	651	70	1097	1300–1500	0.249	0–100	0.283	20–300	0.376	0–100	4.4611	20	0.0040	20
Manganese...	54.93	7.2	20	1260	....	1900	1044	0.1211	20–100	0.228	0–100	....	......	5.0±	......	....	....

*Where the temperature is not given, ordinary temperature is understood.

TABLE III. PHYSICAL PROPERTIES OF METALS (*Continued*)

Name	Atomic Weight	Density		Melting Point, deg cent	Latent Heat of Fusion, g-cal per g	Boiling Point, deg cent	Latent Heat of Vaporization, g-cal per g	Specific Heat		Thermal Coefficient of Linear Expansion		Thermal Conductivity		Electrical Resistivity		Temp Coefficient of Resistivity	
		G per cu cm	Temp,* deg cent					G-cal per g or Btu per lb	Temp, deg cent	$\times 10^4$ per deg cent	Temp, deg cent	G-cal per sec per sq cm per deg cent per cm	Temp, deg cent	Microhm-cm	Temp,* deg cent	Temp Coefficient	Temp, deg cent
Mercury	200.61	13.546	20	−38.87	2.776	356.9	71	0.0333	17	†	...	0.0148	0	95.783	20	0.00089	20
Molybdenum	96.0	10.2	...	2620	...	3700	176.8	0.0589	0	0.049	25–100	0.346	17	5.08	0	0.0047	0–100
Nickel	58.69	8.85	20	1440	73	2900	1010	0.1032	0	0.132	25–100	0.14	0–100	7.8	20	0.00537	20–100
Osmium	190.8	22.48	20	2700	...	5300	350	0.0311	19.98	0.066	40	...	...	9.5	20	...	...
Palladium	106.7	12.0	20	1553	35.93	2200	610	0.0538	0	0.1173	20	0.1683	18	11	20	0.0033	20
Platinum	195.23	21.37	20	1773.5	26.9	4300	637	0.0319	20–100	0.0893	20	0.1664	18	9.83	0	0.003	20
Potassium	39.10	0.870	20	62.3	14.6	760	513	0.177	3.4	0.83	0–50	0.236	0	6.1	0	0.0055	0
Rhodium	102.91	12.44	20	1966	...	>2500	620	0.058	10–97	0.0876	6–21	...	...	5.11	0	0.0043	0
Silver	107.880	10.5	20	960.5	25.9	1950	551.6	0.0557	0	0.197	0–100	1.006	18	1.629	18	0.0038	20
Sodium	22.997	0.9712	20	97.5	27	880	1170	0.283	0	0.622	−190 to −17	0.365	0	4.3	0	0.0054	20
Strontium	87.63	2.60	...	800	...	1150	1045	0.0735	15	...	...	...	...	24.8	20	...	...
Tantalum	181.4	16.6	...	2850	...	>4100	...	0.036	58	0.0655	0–100	0.130	17	15.5	20	0.0031	20
Tellurium	127.5	6.25	20	452	7.3	1390	159	0.0483	15–100	∥0.016 ⊥0.272	20 20	0.0143	45	200,000	19.6	...	...
Thallium	204.39	11.95	20	303.5	...	1650	220	0.0326	20–100	0.302	40	0.093	0	17.6	0	0.0040	0
Thorium	232.12	11.00	17	1845	...	>3000	...	0.0276	0–100	...	...	...	...	18	20	0.0021	20–1800
Tin	118.70	7.29±	...	231.89	14.4	2260	655	0.0548	25	∥0.305 ⊥0.155	20 20	0.155	0	11.5	20	0.0042	20
Titanium	47.90	4.5	18	1800	...	>3000	1320	0.1125	0–100	...	...	...	...	3.0	20	...	...
Tungsten	184.0	19.0±	...	3382	...	5900	1183	0.032	100	0.0444	27	0.476	17	5.50	20	0.0047	0–100
Uranium	238.14	18.7	13	<1850	...	...	...	0.0280	0–98	...	...	...	...	60±18	20	...	...
Vanadium	50.95	5.6	...	1710	...	3000	...	0.1153	0–100								
Zinc	65.38	7.14±	...	419.45	26.6	905±2	426.8	0.0931	20–100	∥0.639 ⊥0.141	20–100 20–100	0.2653	18	5.75	0	0.0037	20
Zirconium	91.22	6.53	...	1900	...	>2900	...	0.066	0–100	...	...	...	...	170±	0	0.00116	−80 to +30

* Where the temperature is not given, ordinary temperature is understood.

† $\frac{1}{v}\frac{dv}{dt} = 182.0 \times 10^{-6}$ at 20 deg cent.

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TABLE IV. PROPERTIES OF SOME ALLOYS

Material	Composition	ρ_{20} Resistivity Microhm-cm 20° C	α_{20} Temp. Coef. 20° C	Thermal Conductivity Ref. to Cu	Specific Gravity	Melt. Point °C
Brass (spring)	72 Cu, 28 Zn	6.0	0.002	0.32	8.6	965
Carbon (brush)	Graphite-carbon	5000 ±	0.0003 ±	0.01	1.5	3600+
Constantan (Advance)	60 Cu, 40 Ni	43	0.00002	0.06	8.9	1300
Manganin	84 Cu, 12 Mn, 4 Ni	47.6	0.00000	0.18	8.5	910
Monel metal	69 Ni, 28 Cu	44.6	0.0020	0.065	8.9	1350
Nichrome	80 Ni, 20 Cr	100.0	0.0001		8.5	1390
Phosphor bronze	96 Cu, 3.75 Sn, 0.25 P	13.7	0.0040	0.16	8.9	1050
Platinum-iridium	90 Pt, 10 Ir	24.6	0.0012	0.07	21.6	
Silver (coin)	90 Ag, 10 Cu	2.4		0.9	10.3	890
Solder (soft)	50 Pb, 50 Sn	17.2			8.8	208
Steel, 4% silicon		62	0.0008	0.05	7.8	1530

TABLE V. Wire Table, Standard Annealed Copper

American Wire Gage (B & S). English Units

Gage No. A.W.G.	Diameter in Mils at 20° C	Cross-section at 20° C: Circular Mils	Cross-section at 20° C: Square Inches	Ohms per 1000 ft * at 20° C (= 68° F)	Pounds per 1000 ft	Feet per Pound	Feet per Ohm † at 20° C (= 68° F)	Ohms per Pound at 20° C (= 68° F)	Pounds per Ohm at 20° C (= 68° F)
0000	460.0	211 600.	0.1662	0.049 01	640.5	1.561	20 400.	0.000 076 52	13 070.
000	409.6	167 800.	.1318	.061 80	507.9	1.968	16 180.	.000 1217	8219.
00	364.8	133 100.	.1045	.077 93	402.8	2.482	12 830.	.000 1935	5169.
0	324.9	105 500.	.082 89	.098 27	319.5	3.130	10 180.	.000 3076	3251.
1	289.3	83 690.	.065 73	.1239	253.3	3.947	8070.	.000 4891	2044.
2	257.6	66 370.	.052 13	.1563	200.9	4.977	6400.	.000 7778	1286.
3	229.4	52 640.	.041 34	.1970	159.3	6.276	5075.	.001 237	808.6
4	204.3	41 740.	.032 78	.2485	126.4	7.914	4025.	.001 966	508.5
5	181.9	33 100.	.026 00	.3133	100.2	9.980	3192.	.003 127	319.8
6	162.0	26 250.	.020 62	.3951	79.46	12.58	2531.	.004 972	201.1
7	144.3	20 820.	.016 35	.4982	63.02	15.87	2007.	.007 905	126.5
8	128.5	16 510.	.012 97	.6282	49.98	20.01	1592.	.012 57	79.55
9	114.4	13 090	.010 28	.7921	39.63	25.23	1262.	.019 99	50.03
10	101.9	10 380.	.008 155	.9989	31.43	31.82	1001.	.031 78	31.47
11	90.74	8234.	.006 467	1.260	24.92	40.12	794.0	.050 53	19.79
12	80.81	6530.	.005 129	1.588	19.77	50.59	629.6	.080 35	12.45
13	71.96	5178.	.004 067	2.003	15.68	63.80	499.3	.1278	7.827
14	64.08	4107.	.003 225	2.525	12.43	80.44	396.0	.2032	4.922
15	57.07	3257.	.002 558	3.184	9.858	101.4	314.0	.3230	3.096
16	50.82	2583.	.002 028	4.016	7.818	127.9	249.0	.5136	1.947
17	45.26	2048.	.001 609	5.064	6.200	161.3	197.5	.8167	1.224
18	40.30	1624.	.001 276	6.385	4.917	203.4	156.6	1.299	0.7700
19	35.89	1288.	.001 012	8.051	3.899	256.5	124.2	2.065	.4843
20	31.96	1022.	.000 802 3	10.15	3.092	323.4	98.50	3.283	.3046
21	28.46	810.1	.000 636 3	12.80	2.452	407.8	78.11	5.221	.1915
22	25.35	642.4	.000 504 6	16.14	1.945	514.2	61.95	8.301	.1205
23	22.57	509.5	.000 400 2	20.36	1.542	648.4	49.13	13.20	.075 76
24	20.10	404.0	.000 317 3	25.67	1.223	817.7	38.96	20.99	.047 65
25	17.90	320.4	.000 251 7	32.37	0.9699	1031.	30.90	33.37	.029 97
26	15.94	254.1	.000 199 6	40.81	.7692	1300.	24.50	53.06	.018 85
27	14.20	201.5	.000 158 3	51.47	.6100	1639.	19.43	84.37	.011 85
28	12.64	159.8	.000 125 5	64.90	.4837	2067.	15.41	134.2	.007 454
29	11.26	126.7	.000 099 53	81.83	.3836	2607.	12.22	213.3	.004 688
30	10.03	100.5	.000 078 94	103.2	.3042	3287.	9.691	339.2	.002 948
31	8.928	79.70	.000 062 60	130.1	.2413	4145.	7.685	539.3	.001 854
32	7.950	63.21	.000 049 64	164.1	.1913	5227.	6.095	857.6	.001 166
33	7.080	50.13	.000 039 37	206.9	.1517	6591.	4.833	1364.	.000 7333
34	6.305	39.75	.000 031 22	260.9	.1203	8310.	3.833	2168.	.000 4612
35	5.615	31.52	.000 024 76	329.0	.095 42	10 480.	3.040	3448.	.000 2901
36	5.000	25.00	.000 019 64	414.8	.075 68	13 210.	2.411	5482.	.000 1824
37	4.453	19.83	.000 015 57	523.1	.060 01	16 660.	1.912	8717.	.000 1147
38	3.965	15.72	.000 012 35	659.6	.047 59	21 010.	1.516	13 860.	.000 072 15
39	3.531	12.47	.000 009 793	831.8	.037 74	26 500.	1.202	22 040.	.000 045 38
40	3.145	9.888	.000 007 766	1049.	.029 93	33 410.	0.9534	35 040.	.000 028 54

*** Resistance at the stated temperatures of a wire whose length is 1000 ft at 20 deg cent.**
† Length at 20 deg cent of a wire whose resistance is 1 ohm at the stated temperatures.

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TABLE VI. ALLOWABLE CURRENT-CARRYING CAPACITIES OF COPPER WIRES *

(National Electrical Code)

Gage No. A.W.G.	Diameter of Solid Wires, mils	Area, cir mils	Rubber Insulation, amp	Varnished Cambric Insulation, amp	Other Insulation, amp
18	40.3	1,624	3	....	5 †
16	50.8	2,583	6	....	10 †
14	64.1	4,107	15	18	20
12	80.8	6,530	20	25	30
10	101.9	10,380	25	30	35
8	128.5	16,510	35	40	50
6	162.0	26,250	50	60	70
5	181.9	33,100	55	65	80
4	204.3	41,740	70	85	90
3	229.4	52,630	80	95	100
2	257.6	66,370	90	110	125
1	289.3	83,690	100	120	150
0	325.0	105,500	125	150	200
00	364.8	133,100	150	180	225
000	409.6	167,800	175	210	275
....	....	200,000	200	240	300
0000	460.0	211,600	225	270	325
....	....	250,000	250	300	350
....	....	300,000	275	330	400
....	....	350,000	300	360	450
....	....	400,000	325	390	500
....	....	500,000	400	480	600
....	....	600,000	450	540	680
....	....	700,000	500	600	760
....	....	750,000	525	630	800
....	....	800,000	550	660	840
....	....	900,000	600	720	920
....	....	1,000,000	650	780	1,000
....	....	1,100,000	690	830	1,080
....	....	1,200,000	730	880	1,150
....	....	1,300,000	770	920	1,220
....	....	1,400,000	810	970	1,290
....	....	1,500,000	850	1,020	1,360
....	....	1,600,000	890	1,070	1,430
....	....	1,700,000	930	1,120	1,490
....	....	1,800,000	970	1,160	1,550
....	....	1,900,000	1,010	1,210	1,610
....	....	2,000,000	1,050	1,260	1,670

* Copper wires and cables of 98 per cent conductivity. For aluminum wire the allowable carrying capacities shall be taken as 84 per cent of those given in the table for the respective sizes of copper wire with the same kind of covering.

† The allowable carrying capacities of No. 18 and No. 16 are 10 and 15 amp, respectively, when in cords for portable heaters, types HC and HPD.

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APPENDIX II

Determinants for Algebra

In algebra, when a set of simultaneous equations is entirely *linear*, determinants may be used instead of the usual processes of elimination, substitution, and the like for their solution. For a set of three or more equations, the certainty of progress with determinants is usually preferable to a gamble with the often elusive elimination process. The successful use of determinants requires no particular skill or ingenuity. For one familiar with algebraic symbolism, it is a game with rules which are scarcely as difficult as those of many popular games of the day (and night). Because reason plays no part in this game, strict adherence to the rules is an absolute necessity.

Consider a set of four general linear simultaneous equations as follows.

$$a_1x + b_1y + c_1z + d_1w = m_1$$

$$a_2x + b_2y + c_2z + d_2w = m_2$$

$$a_3x + b_3y + c_3z + d_3w = m_3$$

$$a_4x + b_4y + c_4z + d_4w = m_4$$

All coefficients (constant) of x, y, z, w are called ***elements*** and are arranged in a grid of ***rows*** and ***columns*** exactly in the order in which they appear above. This arrangement is called a ***determinant*** (D) and is said to be of an ***order*** numerically equal to the number of rows or columns; the number of columns or rows in this case being four, the determinant is said to be of the fourth order.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

For each unknown to be evaluated, such as y, another determinant D_y is formed exactly like D except that the constants m_1, etc., are substituted for the coefficients b_1, etc., of the unknown being evaluated.

$$D_y = \begin{vmatrix} a_1 & m_1 & c_1 & d_1 \\ a_2 & m_2 & c_2 & d_2 \\ a_3 & m_3 & c_3 & d_3 \\ a_4 & m_4 & c_4 & d_4 \end{vmatrix}$$

Each determinant is capable of evaluation, or at least of *reduction*, to a sum of algebraic terms consisting of simple multiples of the elements of the determinant. The value of each unknown, such as y, is the ratio of D_y to D, $y = \frac{D_y}{D}\cdot$ Likewise $x = \frac{D_x}{D}$, etc., where D is the same for all, and D_y, D_x, etc., are formed as for y above.

SOLUTION OF THE DETERMINANT

Solution of a determinant proceeds by expanding it into a sum of determinants of progressively lower order until the second (or possibly third) order is reached, when expansion into a sum of simple algebraic terms becomes possible.

Consider first the determinant D. Select one column, say the first, and set down the first element from this column as a coefficient or multiplier of a new determinant called a ***minor***—a ***first*** minor for the ***first*** element. This new determinant or minor simply consists of what is left of the original determinant after removing both the column and row from which the first element was taken.

$$D = a_1 \begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ \cancel{a_2} & b_2 & c_2 & d_2 \\ \cancel{a_3} & b_3 & c_3 & d_3 \\ \cancel{a_4} & b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} \cancel{a_1} & b_1 & c_1 & d_1 \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} & \cancel{d_2} \\ \cancel{a_3} & b_3 & c_3 & d_3 \\ \cancel{a_4} & b_4 & c_4 & d_4 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} \cancel{a_1} & b_1 & c_1 & d_1 \\ \cancel{a_2} & b_2 & c_2 & d_2 \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} & \cancel{d_3} \\ \cancel{a_4} & b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} \cancel{a_1} & b_1 & c_1 & d_1 \\ \cancel{a_2} & b_2 & c_2 & d_2 \\ \cancel{a_3} & b_3 & c_3 & d_3 \\ \cancel{a_4} & \cancel{b_4} & \cancel{c_4} & \cancel{d_4} \end{vmatrix}$$

This is repeated for each element in the chosen column only, and the sum of these is an expansion of D. It is to be noted that the sign of the elements in *odd*-numbered rows a_1 and a_3 is *plus* while the elements in *even*-numbered rows a_2 and a_4 are *minus* **as coefficients** of their respective minors in the above

As a *check*, the coefficients of successive minors should *alternate* in sign as illustrated above.

The process is performed for **only one** column but some column other than the first may be chosen provided it is recognized that the sign of an element used as a coefficient (of its minor) is affected not only by the number (odd or even) of the row from which it is taken but *also* by the number of the *column* in the same way (odd $= +$, even $= -$). For example, if the d-column (fourth) were chosen instead of the a-column, the element d_3 in row *3* and column *4* would be $(+\ -)\ d_3 = -d_3$ as a coefficient of its minor.

A second expansion reduces the determinants to a second-order sum by repeating the same process just used.

$$D = a_1b_2\begin{vmatrix} \cancel{b_2} & \cancel{c_2} & \cancel{d_2} \\ \cancel{b_3} & c_3 & d_3 \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} - a_1b_3\begin{vmatrix} \cancel{b_2} & c_2 & d_2 \\ \cancel{b_3} & \cancel{c_3} & \cancel{d_3} \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} + a_1b_4\begin{vmatrix} \cancel{b_2} & c_2 & d_2 \\ \cancel{b_3} & c_3 & d_3 \\ \cancel{b_4} & \cancel{c_4} & \cancel{d_4} \end{vmatrix}$$

$$- a_2b_1\begin{vmatrix} \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ \cancel{b_3} & c_3 & d_3 \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} + a_2b_3\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_3} & \cancel{c_3} & \cancel{d_3} \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} - a_2b_4\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_3} & c_3 & d_3 \\ \cancel{b_4} & \cancel{c_4} & \cancel{d_4} \end{vmatrix}$$

$$+ a_3b_1\begin{vmatrix} \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ \cancel{b_2} & c_2 & d_2 \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} - a_3b_2\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_2} & \cancel{c_2} & \cancel{d_2} \\ \cancel{b_4} & c_4 & d_4 \end{vmatrix} + a_3b_4\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_2} & c_2 & d_2 \\ \cancel{b_4} & \cancel{c_4} & \cancel{d_4} \end{vmatrix}$$

$$- a_4b_1\begin{vmatrix} \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ \cancel{b_2} & c_2 & d_2 \\ \cancel{b_3} & c_3 & d_3 \end{vmatrix} + a_4b_2\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_2} & \cancel{c_2} & \cancel{d_2} \\ \cancel{b_3} & c_3 & d_3 \end{vmatrix} - a_4b_3\begin{vmatrix} \cancel{b_1} & c_1 & d_1 \\ \cancel{b_2} & c_2 & d_2 \\ \cancel{b_3} & \cancel{c_3} & \cancel{d_3} \end{vmatrix}$$

The final expansion is performed for each determinant by taking the sum of the products of the coefficient and the elements of each *diagonal;* down to the right giving a plus sign to the product of the elements, whereas up to the right gives a minus sign to the product of these elements.

$$D = (a_1b_2c_3d_4 - a_1b_2c_4d_3) - (a_1b_3c_2d_4 - a_1b_3c_4d_2) + (a_1b_4c_2d_3 - a_1b_4c_3d_2)\cdots, \text{etc.}$$

It is rare that none of the elements of the determinant is zero or unity. Usually several are zero or may be made so by aid of certain permissible modifications of the determinant. The expansion is then not so lengthy as implied in the above. In particular, it is helpful to select a column containing the most elements of value zero as the column of elements to be used as coefficients of their minors in the expansion process.

Operations on Determinants

In algebra several operations may be performed on equations without invalidating the equality. Such processes may expedite the solution of the equations.

Similarly for determinants, several operations are valid and may be used to facilitate an expansion or solution of the determinants. Some of the most useful of these are as follows.

1. All rows may be interchanged with all columns, but a *change of sign* will occur if the move is made as in military drill so that all elements execute either a "left face" or "right face." The sign is *unchanged* for an "about face" or for an exchange of column and row 1, column and row 2, etc., keeping the elements in the same sequence in the new row as in the old column and vice versa.

Example.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} = + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(given) (left face) (row vs. column)

2. As a corollary to (1) all operations applicable to *columns* are equally valid for *rows*. The rules to follow will, for brevity, be stated for *columns* with this understanding.

3. Any two columns may be interchanged with only a change in the *sign* of the determinant.

Example.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

4. *All* the elements of any column may be increased or decreased by the amount of the corresponding elements in any other column or by any fixed multiple of these respective amounts.

Example.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} (a_1 \pm kb_1) & b_1 & c_1 \\ (a_2 \pm kb_2) & b_2 & c_2 \\ (a_3 \pm kb_3) & b_3 & c_3 \end{vmatrix}$$

5. Any *factor* common to all the *elements* of any one column may be considered as a *factor* of the *determinant* and be removed to that position or vice versa.

Example.

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

6. When each element of a given column comprises a sum of two terms, the determinant may be expanded into a sum of two new determinants, each identical with the old except for the given column which is split so that, in each new determinant, each element of this column contains but one of the terms of the corresponding original element.

The above, of course, may be extended to more than two terms, either by repetition or in one operation.

Example.

$$\begin{vmatrix} (a_1 + d_1) & b_1 & c_1 \\ (a_2 + d_2) & b_2 & c_2 \\ (a_3 + d_3) & b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

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APPENDIX III

Concerning Computation

1. A Classification of Quantities. The numerical quantities with which engineers are concerned may be classified into two categories which, although distinctly different, are not in the beginning always clearly recognized to be so, namely,

(1) Exact quantities.

(2) Inexact or approximate quantities.

To classify a given quantity it is necessary to have sufficient information about it. A straight count of discrete objects, correctly executed, is an exact number, e.g., an inventory of 184 spools of wire. An average, however, is likely to be an inexact number, e.g., 184 spools as an average of 11 groups totalling 2025 is inexact, and even if carried to 184.1 or further it is still inexact. One of these spools may be labeled either 1000 ft or x pounds; neither of these can be exact because they can only be *measured* and not actually counted. In general a measurement of length, weight, volume, temperature, humidity, voltage, power, etc., is *never* an exact number. The allowable error depends on the purpose of the measurement and the means for making it. We do not expect a ton of coal to be weighed within an ounce of error but we may expect a pound of mercury to be weighed with less error, and a pound of platinum with far less error because of the higher monetary consideration per ounce. If we weigh a pound of charcoal preliminary to a fuel analysis of it of course we require a different standard of error than if we are to use it for a steak roast.

We have now observed that whether a number is whole like 184 or a decimal like 184.1 is no criterion of exactness; either number may or may not be exact depending on how it is used. Let us examine some other kinds of numbers. Consider an incommensurate quantity, like π; although 3.14159265 is inexact, π itself represents an exact quantity, i.e., the ratio of the circumference to the diameter of any circle. The decimal 0.6667, when used to express $\frac{2}{3}$ is inexact although $\frac{2}{3}$ itself may be an exact quantity. Another situation is encountered when we wish to make a given quantity n times as large (not n times *larger*, which is $n + 1$ times *as large*); it is presumed that n is an exact number although the given quantity, and n times the given quantity, may or may not be exact quantities.

When numbers are exact, like the quantities of merchandise in an inventory, the usual arithmetic operations of addition, subtraction, multiplication, and sometimes division give exact numbers free from arithmetic uncertainties. The quantities which mostly concern engineers are quantities measured not by counting discrete indivisible units like spools or motors but by reading values

marked on a scale like inches, centimeters, volts, amperes, miles per hour, and seconds, and we must understand how to express and to deal with these inexact numbers at least for the ordinary operations of arithmetic.

*2. **Representation of Inexact Numbers.*** It has been found convenient to express the uncertainty of an inexact number as a fraction or *per cent* of the number. When, as usual, the number is equally likely to fall over or under the true value, the per cent is given the sign $\pm$, e.g., 300 ± 1 per cent means that the true value lies between $(300)(1.01) = 303$ and $(300)(0.99) = 297$. The means for determining these percentages cannot be fully treated here. Manufacturers of the better grades of measuring instruments designate the inherent precision of their instruments as "accurate to x per cent." Of course, this presumes that the instrument receives adequate care and maintenance. If the number read from the instrument is to have only this per cent uncertainty, it is further necessary that the reading be done with due care, e.g., to avoid parallax by utilizing the built-in mirror or even by utilizing reflections in the glass face over the scale, and that an instrument of proper *range* be properly *applied* to obtain a true measure of the desired quantity.

*3. **Probability.*** It is usual to reduce the uncertainty of a measurement by *repetition*, i.e., to invoke the services of *probability*. The laws of chance are known to favor the occurrence of readings nearer the true value. The most generally accepted law of distribution is that of Gauss and Laplace.

If we stack each group of duplicate readings and arrange these stacks in sequence horizontally from left to right along a linear scale of their values, a smooth line drawn over the top of this arrangement will approximate the Gauss-Laplace curve of Fig. 1. This distribution is expressed by

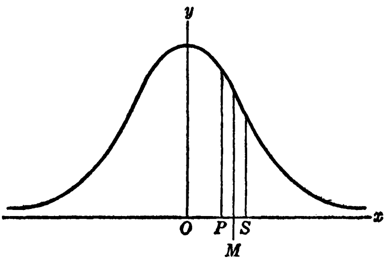

FIG. 1.

$$y = A\epsilon^{-bx^2}$$

where x is scaled with zero value at the y ordinate shown, so that the values of x are the amounts by which the readings deviate from the average or "most probable" value.

The errors x (deviations from the most probable value) are characterized in three ways so often encountered that they bear recognition here.

(1) The *probable error* (p.e.). That magnitude of error (OP Fig. 1) above and below which the number of errors is equal, i.e., that error which will not be exceeded by 50 per cent of the readings.

(2) The *mean deviation* (m.d.). The arithmetic average of all the deviations or errors (OM Fig. 1).

(3) The *standard deviation* (s.d.). The root mean square of all the deviations or errors (OS Fig. 1).

These quantities are related to one another by the simple ratios:

$$\text{p.e.} = 0.85 \text{ m.d.} = 0.67 \text{ s.d.}$$

It is interesting to note that OP of (1) divides the first quadrant area in halves, that OM passes through the center of gravity of this area, and that OS intersects the point of inflection in the curve.

It is evident that the higher the accuracy of the readings the less will each of these quantities be. When a quantity is written 300 ± 3 it is understood that ± 3 is the probable error, so that for our previously considered 300 ± 1 per cent, the 1 per cent is the *per cent probable error.*

Obviously no amount of repetition can eliminate those errors which, knowingly or otherwise, are consistently repeated. Repetition cannot correct the indication of instruments with bent pointers, maladjusted zero indication, or improper application. Neither can it correct for errors in reading scale divisions or inattention to parallax except (possibly) when the several readings are each obtained from a different operator. It is unnecessary here to become involved in the statistical processes which concern the various aspects of this procedure, but it is important to understand in a general way the forces which influence the uncertainty of inexact quantities.

4. Operations with Inexact Numbers. It is important to know how to handle inexact numbers in computation so that the uncertainty or probable error in the end result is known. It is often no less important to perform the converse operation of determining how accurately each of several quantities involved in a test or research must be measured to obtain the desired accuracy in the final result. Otherwise much time may be wasted by making certain measurements either more or less accurately than is useful.

For this purpose it is necessary first to examine the total effect on a quantity which the errors of its components produce. Let us consider a quantity

$$x = a + a = 2a$$

where

$$a = 23.4 \pm 0.2$$

It seems quite clear that the result is

$$x = 2(23.4 \pm 0.2) = 46.8 \pm 0.4$$

and that the *per cent* probable error in x is the same as that in a.

Now let us consider a different equation $y = a + b$, where a and b are not basically equal by any known law, but where it happens that a given situation gives data $a = 23.4 \pm 0.2$ and $b = 23.4 \pm 0.2$ so that we write

$$y = (23.4 \pm 0.2) + (23.4 \pm 0.2)$$

For exact quantities there would appear to be no distinction between $x = 2 \times 23.4 = 46.8$ and $y = 23.4 + 23.4 = 46.8$. For inexact quantities, however, there is an important distinction; since a and b of $y = a + b$ are not actually identical components, any particular value (reading) of a with, say, a *plus* error is as likely as not to be accompanied by a value of b with *minus* error. On the

average this "seesawing" of the errors in a and b will give a value of y with less probable error than a alone can give for x. In fact, it is by much the same process that we expect *two* (or more) readings, say a and b of a quantity, to give *less* probable error than would the *one* reading a. It remains to provide some suitable means for combining the errors in a and b to give a value less than their plain arithmetic sum. The means commonly used derives from what is known as the *method of least squares*. It seems inadvisable here to go into this subject beyond the finding that the most probable total error is given by the square root of the sum of the squares of the errors produced in the quantity by its components, each acting independently of the others, i.e.,

$$\Delta y = \sqrt{(\Delta y)_a^2 + (\Delta y)_b^2 + \cdots (\Delta y)_n^2}$$

where $(\Delta y)_a$, $(\Delta y)_b$, etc., are the respective errors produced in y by its components a, b, etc. Whether the components are multiplied, divided, summed, or combined by some other process ***is of no concern*** in this relation.

For the example, we now proceed to find

$$\Delta y = \sqrt{0.2^2 + 0.2^2} = 0.28$$

and write $y = 46.8 \pm 0.3$.

Where there are but two components the total probable error, of course, is represented by the hypotenuse of a right triangle having legs equal to the respective probable error components. It is clear in such case that the total probable error so obtained really is less than the arithmetic sum of the component probable errors. To include more than two or three components in this geometric concept of course becomes difficult but the effect is reasonably apparent.

Let us consider a more general example of this kind where there are three components, as follows

$$y = a + b + c$$

where

$$a = 461.6 \pm 2.3$$

$$b = 240 \pm 0$$

$$c = 6.8 \pm 0.68$$

The error in y is

$$\Delta y = \sqrt{(\Delta y)_a^2 + (\Delta y)_b^2 + (\Delta y)_c^2}$$

$$= \sqrt{(2.3)^2 + (0)^2 + (0.68)^2} = 2.4$$

And

$$y = (461.6 + 240 + 6.8) \pm \Delta y$$

$$= 708.4 \pm 2.4$$

Let us examine the *per cent* errors

$$a = 461.6 \pm 0.5\%$$

$$b = 240 \pm 0$$

$$c = 6.8 \pm 10\%$$

$$y = 708.4 \pm 0.34\%$$

We find that the influence of the exact component b is sufficient to give y a higher per cent accuracy than a has. The effect of c, which has very poor accuracy, is negligible because the value of c is less than 1 per cent of the total quantity y. Note that measurement of c to any higher accuracy would not be worth while because the effect on Δy is already very small.

The Difference of Inexact Numbers. The same relation $\Delta y = \sqrt{(\Delta y)_a^2 + (\Delta y)_b^2}$ applies to a difference as well as to a sum of a and b, still using the sum and not the difference of the squared component errors.* It will be found especially useful to observe the result of taking the *small* difference between two compartively *large* quantities. Let us consider $y = a - b$ where

$$a = 5280 \pm 5.3 \text{ or } \pm 0.1\%$$

$$b = 5200 \pm 5.2 \text{ or } \pm 0.1\%$$

Then

$$\Delta y = \sqrt{5.3^2 + 5.2^2} = 7.6$$

and

$$y = 80 \pm 7.6 \text{ or } \pm 10\%$$

Note that in spite of the high accuracy of a and b the difference y is only $1/100$ as accurate. Evidently it would be preferable to avoid any situation which requires determination of $y = a - b$ by measurement of a and b when they have values of comparable magnitude.

The Product of Inexact Numbers. When the quantity is a product of factor components rather than a sum, the effect of their uncertainties is best handled in terms of the fractional or *per cent errors*. The per cent probable error in a quantity is then the square root of the sum of the squares of the per cent probable errors of the individual factors of this quantity. When any factor has an

* The difference rather than the sum of the squares occurs only in the converse operation where a desired error in the quantity y, and in all but one of its components, are given to find the allowable error in the one component. Referring to the example $y = a + b + c$ where now we are given $\Delta y = \pm 2.4$, $\Delta b = 0$, and $\Delta c = \pm 0.68$ to find Δa we write:

$$\sqrt{(\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2} = \Delta y$$

or

$$\Delta a = \sqrt{(\Delta y)^2 - (\Delta b)^2 - (\Delta c)^2}$$

$$= \sqrt{(2.4)^2 - (0)^2 - (0.68)^2} = \pm 2.3$$

exponent the effect of its uncertainty is directly in proportion to the exponent. Let us consider

$$y = \pi x^3 z^{1/2}$$

where

$$x = 4.08 \pm 0.02 \text{ or } \pm 0.50\%$$

$$z = 59.3 \pm 0.6 \text{ or } \pm 1.0\%$$

$$\left(\frac{\Delta y}{y}\right)_x = 3 \times 0.50 = 1.50\%$$

$$\left(\frac{\Delta y}{y}\right)_z = \tfrac{1}{2} \times 1.0 = 0.5\%$$

$$\frac{\Delta y}{y} = \sqrt{\left(\frac{\Delta y}{y}\right)_x^2 + \left(\frac{\Delta y}{y}\right)_z^2} = \sqrt{1.50^2 + 0.5^2} = 1.6\%$$

Then $y = \pi(4.08)^3(59.3)^{1/2} \pm 1.6\% = 1643 \pm 1.6\%$ or ± 26.

The Quotient of Inexact Numbers. The foregoing discussion of products involving factors with exponents other than unity is not confined to *positive* exponents. Because a factor with a negative exponent is in effect a divisor having the numerically same positive exponent, the quotient of inexact numbers requires no special consideration. The minus sign of an exponent, of course, cannot cause reduction in the total error because it is lost in taking the *squares* of the component per cent errors; it is still the actual arithmetic *sum* of these squares that gives the square of the total per cent error.

The Converse Operations. When we wish to determine a quantity with a given accuracy by measurement of its components we must establish the accuracy with which each component must (and can) be measured. We usually approach this problem by assuming that we shall (and can) measure each component with *equal* accuracy. We have observed for $y = f(a, b, c \cdots n)$

that

$$\Delta y = \sqrt{(\Delta y)_a^2 + (\Delta y)_b^2 + (\Delta y)_c^2 \cdots (\Delta y)_n^2}$$

Now for n components equal accuracy requires that

$$(\Delta y)_a = (\Delta y)_b = (\Delta y)_c \cdots = (\Delta y)_n$$

we readily find

$$\sqrt{n(\Delta y)_n^2} = \Delta y$$

or

$$(\Delta y)_n = \frac{\Delta y}{\sqrt{n}}$$

For example, consider the previous equation

$$y = \pi x^3 z^{1/2}$$

If we wish y to have a probable error of 1 per cent

$$\left(\frac{\Delta y}{y}\right)_x = \left(\frac{\Delta y}{y}\right)_z = \frac{1}{\sqrt{2}} = 0.71\%$$

Then, since

$$\left(\frac{\Delta y}{y}\right)_x = 3\frac{\Delta x}{x}$$

$$\frac{\Delta x}{x} = \frac{1}{3} \times 0.71 = 0.24\%$$

and since

$$\left(\frac{\Delta y}{y}\right)_z = \frac{1}{2}\frac{\Delta z}{z}$$

$$\frac{\Delta z}{z} = 2 \times 0.71 = 1.4\%$$

If it should be found impractical to measure x to within ± 0.24 per cent, while z is readily measured to within ± 1.4 per cent, we would then modify our assumed equality of per cent errors to balance with the difficulty of measurement of the quantities x and z.

If the components are measurable to the same per cent accuracy with equal ease, we would probably prefer to have the same per cent accuracy *for each component* instead of the same per cent *effect by* each component. Then for our example:

$$\frac{\Delta y}{y} = \sqrt{\left(3\frac{\Delta x}{x}\right)^2 + \left(\frac{1}{2}\frac{\Delta z}{z}\right)^2}$$

where

$$\frac{\Delta x}{x} = \frac{\Delta z}{z} = r$$

so that

$$\frac{\Delta y}{y} = \sqrt{(3r)^2 + (\tfrac{1}{2}r)^2} = \sqrt{(9.25)^2}$$

and

$$r = \frac{\Delta y/y}{\sqrt{9.25}}$$

or

$$r = \frac{1}{\sqrt{9.25}} = 0.33\%$$

and each component, x and z, must be accurate within $\pm\frac{1}{3}$ per cent to provide ± 1 per cent accuracy for y.

Summary of Observations on Operations with Inexact Quantities. When a quantity y is comprised of sums or differences of terms as well as products or quotients or powers it is important to observe that while the errors in the *factors* of

a product, quotient, or power term *must be in per cent,* the errors of the terms to be *added* or *subtracted* must *not* be in per cent; it is necessary, in general, to convert errors to and from the per cent form according to the nature of the operation.

We have observed that the per cent error in the *sum* of inexact numbers lies between that of the best and the worst of the components—a weighted *mean* value. We have also observed that the per cent error in the *difference* of inexact numbers, as we might expect of the converse operation, is in no sense an average and *may greatly exceed* the per cent error of either component quantity. For products, quotients, and powers not less than unity but either plus or minus, the per cent error is never less than that of the worst factor. Only where powers numerically less than unity are concerned can the per cent error in the resultant quantity be less than that of the worst factor; even then it *may* not be less, depending on the quality, kind, and number of the other factors.

It is important not to memorize these statements as such but to check them off against the reader's grasp of the preceding discussion.

5. Significant Figures. It is by no means always necessary or desirable to represent inexact quantities as completely as does the appendage of probable error. We commonly express an inexact quantity by a number having from left to right as many digits as are certain plus another which is uncertain or "doubtful," e.g., 213.4 is usually taken to mean 213.4 ± 0.1 so that the 4 is doubtful and might possibly be either 3 or 5 but not 2 or 6.

When we have a number like 2500 we do not know how many, if any, of the zeros are "significant," i.e., have meaning other than as mere fillers to locate the decimal point; we are uninformed whether this may mean exactly 2500, 2500 ± 1, 2500 ± 10, or 2500 ± 100. The nature or source of the number usually indicates whether or not it is exact. If inexact it can best be represented as 2500, 250.0×10, or 25.00×10^2 according to the respective possibilities just enumerated. When the number is 2500.0 it is clearly understood to mean 2500.0 ± 0.1. Note that significant figures may be either exact or inexact.

6. Operations with Significant Figures. It is customary to perform arithmetic operations with significant figures according to certain rules which it must be understood can give only approximate results because they cannot be other than a poor substitute for the per cent probability procedure previously discussed.

For addition consider

$$\begin{array}{r} 274.3 \\ 53.56 \\ 708 \\ \hline 1035.86 \end{array}$$

Since we do not know whether the 3 in the first number may not be either 2 or 4 it is misleading to leave the 6 in the sum; we do not know whether the 8 there may not be 7 or 9. The number should be written 1035.9. This is true only if we know that 708 is either an exact number or at least 708.0. If the latter, it

should be so written. If it is correct "as is," and is not an exact number, we can write only 1036 for the sum. It is usually considered best to conserve time by eliminating before addition the digits in each number which are going to be doubtful in the sum, making the last digit one higher if the adjacent figure eliminated is 5 or more, thus:

$$\begin{array}{r} 274 \\ 54 \\ 708 \\ \hline 1036 \end{array}$$

The author believes, however, that it is justifiable and advisable to eliminate *one less* than this number *before* addition and then to eliminate this digit only in the result.

For multiplication consider

$$274.3 \times 5.6$$

$$\begin{array}{r} 274.3 \\ 5.6 \\ \hline 164.58 \\ 1371.5 \\ \hline 1536.08 \end{array}$$

The uncertainty of the 3 in one factor and the 6 in the other factor evidently affect both the 8 and the 0 in the result and possibly the 6 also. To clarify the situation let us make a test with the per cent error method we have already used. With each error assumed ± 0.1, the per cent errors are $0.1/274.3 = 0.036$ per cent and $0.1/5.6 = 1.8$ per cent. The product then should have error $\sqrt{0.036^2 + 1.8^2} = 1.8$ per cent. The numerical error is $(0.018)(1536) = 28$ and the true value lies between 1508 and 1564. It becomes clear that we are being optimistic to write even 1540 or better, 15.4×10^2. The generally accepted rule is that the product should retain no more digits than the least contained in any of the component factors. Some judgment must be exercised in these matters, e.g., if one factor is 9.9 and the other 121.1 we certainly should retain three digits in the product rather than two. Possibly any number with the first digit 5 or more should be rated for this purpose as having one more significant figure than the actual. On this basis we would be justified in retaining the 4 in the above 1540; otherwise only 1500, which is rather severe pruning.

*7. **Slide Rule Accuracy.*** The basic slide rule is an instrument for computing the products and quotients of numbers by respectively adding and subtracting physical lengths which are made proportional to the logarithms of the numbers with which the lengths are labeled.

It is popularly understood that slide rule computations are highly approximate and consequently inferior to the results of long hand arithmetic or of computing machines of the keyboard type which operate on the component digits of each

number. When dealing with exact numbers, as in banking and accounting, this is true in proportion to the number of digits contained in the largest number.

In engineering computations, however, exact numbers are the exception; numbers commonly represent measured or scaled quantities which contain as many quantitative digits or *significant figures* as the quality of measuring apparatus, care in operation, and closeness of reading may dictate. If the size and quality of the slide rule permit handling all the significant figures obtained in the data to be computed, the results obtained need not be more approximate than the original data. In accord with the previous study here of "Operations with Significant Figures," it becomes apparent that more exact methods of computation would not only be a waste of time but could actually misrepresent the accuracy of the result.

It is misleading to rate the accuracy of slide rule scales on the basis of significant figures. True as it may be that the number of significant figures which can be handled at one end of the scale differs from that at the other end, this is not significant. We must evaluate the scale accuracies in terms of the more basic numbers expressed with their probable error.

Let us consider first the accuracy of the square (A, B) scales as compared with the C or D scales. We have found that the per cent probable error for the square of a quantity is twice that of the quantity; that the per cent probable error is directly proportional to *any* power to which the quantity is raised. Now since the square scales are exactly *half* as long as the CD-scales the reading of the square scale can be but *half* as accurate as the reading of the CD-scales, i.e., exactly *in harmony* with the behavior of the per cent probable error. A similar relation, of course, exists for the cube scale, so that these scales automatically provide readings with the proper accuracies.

As for the relative accuracy of the various parts of any one scale, such as the D-scale, let us observe that the basic limitation is the actual distance, say in thousandths of an inch, which we can distinguish, i.e., the probable error in mils (milli-inches) which we will make in setting or reading the slide or hairline of the rule. This error in *mils* is the same, of course, for any part of the rule.

For example, let us presume that this amount (exaggerated) is equal to that of the space between the left index of the D-scale and the first division of the scale (101). The per cent error in reading 100, i.e., 100 ± 1, would then be $\frac{1}{100}$ or 1 per cent. This same space in mils at the scale number 5 would make 500 doubtful (500 ± 5) by $\frac{5}{500}$ or also 1 per cent. At the right-hand index the same space in mils makes 1000 doubtful (1000 ± 10) by $\frac{10}{1000}$ which again is 1 per cent. It is thus clear that, for a given probable error in discernment of a scale division (in mils), the *per cent* probable error is the same throughout the scale length. Because we are concerned not with the addition of the scaled numbers on a slide rule but with their products and quotients it is the *per cent* errors which directly participate, as discussed under the earlier treatment of this subject.

It follows that the slide rule, far from being the imperfect "guess stick" which it is sometimes unwittingly thought to be, is a well-balanced instrument for

computing the products, quotients, powers, and roots of inexact numbers which have probable errors commensurate with the length of the given rule.

It is well to observe here that the foregoing does *not* apply necessarily to the scales which are usually included for other purposes on the slide rule; the trigonometric scales, for example, are by no means uniform in the per cent error of the various parts of their scales.

Note should be made that operations performed with log tables having the proper number of places (significant figures) to match the per cent error of the most inexact item in the data will give results with the correct number of significant figures in the same way as does the slide rule.

*8. **The CI-Scale.*** The chronic slide rule user will be well repaid in learning to use the inverted or *CI*-scale with proficiency. The number of motions is thereby reduced and the accuracy of the result improved. This is because the number of settings is reduced, especially by avoiding the substitution of slide index for hairline at a given scale value. Consider the following product:

$$3.82 \times 2.46 \times 7.35 = 69.1$$

Without the *CI*-scale this might proceed as follows.

A. (1) Move left of *C*-scale to 3.82 of *D*-scale.
(2) Move hairline to 2.46 of *C*-scale.
(3) Move right index under hairline.
(4) Move hairline over 7.35 on *C*-scale.
(5) Read 69.1 under hairline on *D*-scale.

Using the *CI*-scale this is shortened to:

B. (1) Move left index of *C*-scale to 3.82 of *D*-scale.
(2) Move hairline to 2.46 of *C*-scale.
(3) Move 7.35 of *CI*-scale under hairline.
(4) Read 69.1 on *D*-scale at *C*-index.

An alternative procedure uses the *CI*-scale before the *C*-scale, as follows.

C. (1) Move hairline to 3.82 of *D*-scale.
(2) Move 2.46 of *CI*-scale under hairline.
(3) Move hairline to 7.35 of *C*-scale.
(4) Read 69.1 on *D*-scale at *C*-index.

It is clear that there are but three moves instead of four when the *CI*-scale is used. Furthermore, the question of left or right index in moves 1 and 3 of *A* are partially or entirely avoidable by using the *CI*-scale as in *B* and *C*, respectively.

For division the same advantage of fewer moves and readings will be found. If it is assumed that the procedure for multiplication and division with the *C* and *D* scales are well-mastered, the *inverse* scale may best be mastered by remembering that we *operate* it *inversely*, i.e., we *multiply* with the *CI*-scale by moving it as we would to divide with the *C*-scale, and vice versa.

9. Vector Computation on the Slide Rule. The prospective purchaser of a slide rule for electrical engineering is likely to hear about a "vector rule." The only "vector" operations which slide rules perform are readily understood in terms of trigonometry or even of the plane geometry of right triangles. Given two sides A and B of a right triangle, to find the hypotenuse $C = \sqrt{A^2 + B^2}$ and angle $\alpha = \tan^{-1}\frac{B}{A}$, the trigonometric scales on certain rules, including the log-log duplex vector and log-log duplex decitrig, are especially arranged to facilitate solution. The important feature is that the sine scale S and the tangent scale T *both* refer to the D-scale instead of one to the A-scale as do many rules. This determines, regardless of label ("vector" or otherwise) whether or not the rule is equipped to handle "vectors" efficiently.

With the decitrig rule let us compute the following.

$$\sqrt{64^2 + 40^2} = 75.5$$

1. Move the right index on 64 of the D-scale.
2. Move hairline on 40 of the D-scale.
3. Read under hairline 32.0° on the T-scale.
4. Move 32.0° on the S-scale under hairline.
5. Under slide index read 75.5.

The angle 32.0°, of course, is the angle $\alpha = \tan^{-1} {}^{40}\!/_{64}$.

The process evidently does not actually take the root of the sum of the squares but utilizes the equivalent trigonometric procedure of finding

$$\alpha = \tan^{-1}\frac{B}{A}$$

and

$$C = \frac{B}{\sin\alpha}$$

For the inverse computation, finding the root of the *difference* of squares, let us consider:

$$\sqrt{75.5^2 - 40^2} = 64$$

The process is as follows.

1. Move right index on 75.5 of the D-scale.
2. Move hairline on 40 of the D-scale.
3. Read under hairline 32.0° on the S-scale.
4. Move 32.0° on the T-scale under hairline.
5. Under slide index read 64.0.

It is important to observe in these operations that the first move is always to place the index over the *larger* number. Whether the right or left index is to be used depends on whether the smaller number is to the left or the right respectively of the larger number, i.e., so that the S and T scales will be positioned

to have values over the smaller number on the D-scale. In the first example, just considered, the smaller number 40 was to the *left* of the larger number 64 so that the *right* index was used. If, however, the problem were

$$\sqrt{6.4^2 + 40^2} = 40.5$$

the smaller number 6.4 would be to the *right* of the larger number 40, and the *left* index would be required to initiate the following:

1. Move left index on 40 of the D-scale.
2. Move hairline on 6.4 of the D-scale.
3. Read under hairline 9.09° on the T-scale.
4. Move 9.09° on the S-scale under the hairline.
5. Under slide index read 40.5.

The operations that would be required for the inverse problem

$$\sqrt{40.5^2 - 6.4^2} = 40$$

should begin, of course, with the left index on 40.5 and proceed with little difficulty when the examples already given are understood.

10. An Approximate Solution. When in the relation $C = \sqrt{A^2 + B^2}$ it happens that $A \geq 3B$ it is sufficient for most purposes to utilize the first two terms $A + B^2/2A$ of the series expansion $A + \frac{B^2}{2A} - \frac{B^4}{8A^3} + \frac{B^6}{16A^5} \cdots$. Given $\sqrt{50^2 + 15^2}$ we find

$$C \approx 50 + \frac{(15)^2}{2 \times 50} = 50 + \frac{225}{100} = 52.25$$

When A is not an exact number care regarding the number of significant figures permissible in the result must be observed.

For $C = \sqrt{A^2 - B^2}$, when $A \geq 3B$, the process is:

$$C \approx A - \frac{B^2}{2A}$$

The above approximate computations are of accuracy comparable with that of a good 10-in. slide rule.

This is but one case of a more general approximation which is so useful that it merits more than passing mention. It is understood, of course, that a is no more than $\frac{1}{3}$, and preferably $a < 0.1$.

$$(1 \pm a)^m \approx 1 \pm ma$$

The values $m = \frac{1}{2}, 2, -\frac{1}{2}, -1, -2$ are especially common.

It is useful also to remember that

$$(1 \pm a)^{\pm 1}(1 \pm b)^{\pm 1}(1 \pm c)^{\pm 1}, \text{etc.} \approx 1 \pm a \pm b \pm c, \text{etc.}$$

where the sign of a, b, c, etc., in the result is governed not only by its original sign but also by that of the exponent ± 1 with which it is associated.

When two quantities x and y are nearly alike, the geometric mean is approximately the arithmetic mean, as follows.

$$\sqrt{xy} \approx \frac{x + y}{2}$$

11. Slide Rule Evaluation of Exponentials. The owner of a slide rule such as the *K & E* log-log duplex decitrig which has the *LL0* and *LL00* scales should take advantage of them in dealing with the problems in Chapters XV, XVII, and XVIII which involve the evaluation of exponential functions with negative exponents. These scales directly give values of $y = \epsilon^{-x}$ when $0.001 < x < 10$ simply by setting values of x on the *A*-scale and reading y on the *LL0* or *LL00* scales. The left index represents $x = 0.001$ for y on the *LL0* scale, or $x = 0.1$ for y on the *LL00* scale. Of course the right index represents $x = 0.1$ for y on the *LL0* scale or $x = 10$ for y on the *LL00* scale. To find $y = \epsilon^{-1.51}$, for example, set the hairline on 1.51 of the right half of the *A*-scale, and on the *LL00* scale read $y = 0.221$. To find $y = \epsilon^{-0.0151}$ the same setting gives on the *LL0* scale $y = 0.985$.

12. Finding the Decimal Point. In the early stages of slide rule use, the question of locating the decimal point seems very annoying, and an infallible rule of thumb for it seems of great importance. In engineering, it is almost universal to locate the decimal point by an approximate recomputation using only one, or at most two, significant figures. Consider the previously used example.

$$3.82 \times 2.46 \times 7.35 = 69.1$$

This is rewritten with one significant figure for each number, as follows.

$$4 \times 2 \times 7 = 60$$

It is clear that 69.1 is nearer 60 than is either 6.91 or 691. Furthermore, if the slide rule result had come out with the figures 128, it would be clear that a number differing so greatly from 60 must be in error. Thus we not only find the decimal point by this means but also provide a rough check on the result so that gross errors are avoided. Cancellation should be utilized whenever possible. Given the example

$$\frac{3.82 \times 2.46 \times 7.35}{0.427 \times 5.73 \times 137} = 0.206$$

we proceed as follows.

$$\frac{\overset{\cancel{10}}{\cancel{4} \times \cancel{2} \times 7}}{\underset{\substack{\cancel{20} \\ \cancel{10}}}{0.\cancel{4} \times 6 \times \cancel{140}}} = \frac{1}{6} = 0.2$$

The approximate value 0.2 is quite adequate to show that the result cannot be 2.06 or 0.0206, and that a result with figures such as 674 would surely be in error.

Consider now the location of the decimal point for the following example previously discussed.

$$\sqrt{64^2 + 40^2} = 75.5$$

We would readily sense that the result must be of an order of magnitude more than the larger number (64) and less than $\sqrt{2} \times 64$, or, using 1.5 to approximate $\sqrt{2}$, the result must be less than $1.5 \times 64 = 96$. That the result could not be 7.55 or 755 is then obvious. Furthermore, a result with figures such as 528 would clearly be in error.

In like manner we may check that the angle

$$\alpha = \tan^{-1} \frac{40}{64} = 32.0°$$

is a reasonable value. On the slide rule it is easy to make the mistake of reading the *complementary* angle 58.0° instead of the *correct* angle unless the usual red and black color code is entirely clear. Bearing in mind that $\tan 45° = 1$, it is a simple matter to observe that a ratio B/A less than unity, such as our $^{40}/_{64}$ must have an anti-tangent *less* than 45°, and therefore 32.0° is correct here rather than 58.0°.

One of the qualities which distinguish a good engineer is his formation of habits of alertness in devising and applying checks such as the above on his work and that of others which concern him. In time he develops a formidable defense against errors and fallacies in computation and reasoning.

13. ***Ratio and Proportion.*** The slide rule is particularly adapted to the solution of problems in proportion. It is important that such problems be recognized in the course of a computation and that advantage be taken of them in slide rule operation. Consider, for example, the relation

$$E = K\phi S$$

Given $S = 1800$, $\phi = 10{,}000$, $E = 120$ to find E when $S = 1500$ and $\phi = 8000$. The temptation seems to be to solve first for K, substitute this value in the equation and re-solve for E. It is much better to rewrite the equation for the new values and set up a proportion.

$$\frac{E' = K\phi' S'}{E = K\phi S}$$

Observing that K has the same value in each equation the quotient of the two equations is

$$\frac{E'}{E} = \frac{\phi' S'}{\phi S}$$

Solving for E' and substituting values,

$$E' = E\frac{\phi' S'}{\phi S} = 120\,\frac{11{,}000 \times 1500}{10{,}000 \times 1800} = 110$$

The same computation procedure is involved here as would have been required by the "evaluation of K" method. In the above, however, we avoid the actual reading of K and the *resetting* of the K value. That fewer motions and fewer readings attend the proportionality method is evident.

The proportionality method has other advantages. Consider again the relation $E = K\phi S$, but let it be given to find the value of S when ϕ is increased 10 per cent and E is decreased 8.3 per cent, the original value of S being 1800. As before, we write

$$\frac{E'}{E} = \frac{\phi' S'}{\phi S}$$

But now

$$\frac{S'}{S} = \frac{E'}{E} \times \frac{\phi}{\phi'}$$

Substituting

$$\frac{S'}{1800} = \frac{100 - 8.3}{100} \times \frac{100}{100 + 10}$$

$$S' = 1800\frac{91.7}{110} = 1500$$

To find K under these conditions, of course, is impossible and if one is accustomed to proceeding by the K-evaluation method he is, to put it mildly, at a disadvantage. Furthermore, it is always helpful to visualize a relation like the above in terms of proportion: "E is directly proportional to ϕ and to S," or "S is directly proportional to E and inversely to ϕ." Thinking in these terms is far better than simply memorizing an equation and applying it as so much algebra. Thinking in proportionalities, it is readily visualized whether a certain result should be larger or smaller than the original value. This is one of the means by which engineers *think straight* and check against foolish errors.

14. Percentage. The term "per cent" is used in connection with so many phases of everyday life that very often there is a strong tendency for its true meaning and significance to be obscured. We as engineers are especially accustomed to think of our quantities in terms of per cent fully as much as in terms of the actual values of the quantities themselves.

The subject of percentage involves no new principle but is merely an application of the previous study of ratio and proportion. If we set up the equality of ratios, $\frac{A}{B} = \frac{X}{100}$, and solve for X, we find that $X = \frac{A}{B}100$ and that X is the value of the ratio $\frac{A}{B}$ in *per cent.* Per cent is another way of saying *per hundred.* The sign commonly used of course is %. The quantities X per cent, $X\%$, and $\frac{X}{100}$ all represent the same thing, namely, X per hundred or X hundredths.

In the ratio just given, the *number B*, of which some per cent is found, is called the ***base.*** It is the term which is so often either misused or inadvertently over-

looked when working with per cents. If it is always kept in mind that in dealing with per cents we are dealing with *ratios*, of which the denominator is the base (or the reference number), a great deal of difficulty and confusion can be avoided. The *number* A is the *percentage* of the base B obtained by multiplying the base B by the quantity X in per cent.

It is entirely proper that the above proportion, $\frac{A}{B} = \frac{X}{100}$, in terms of X be expressed as a ratio y, i.e., $\frac{A}{B} = \frac{y}{1}$ which on solving becomes $y = \frac{A}{B} \times 1$. In this case instead of calling the quantity "per cent" ($X\%$ or X per hundred) it is called a "per unit" quantity ($y/1$, or y per unit). The symbol sometimes used for this quantity is 1/1. Thus $\frac{1}{20} \times 1.0 = 0.05$ 1/1, and conversely 0.05 1/1 of 20 is $\frac{0.05}{1} \times 20 = 1.0$. In per cent this would be $\frac{1}{20} \times 100 = 5$ per cent, and it is seen that per cents are per units times 100. This method of expressing ratios is just as practical and workable as the one using the name per cent, and it so happens that for certain of our work in electrical engineering it is to be preferred.

The term per cent is often confused when speaking of a given quantity as being increased by some per cent and then decreased by the same per cent. The tendency is to conclude that the final value is the same as the original value. This is wrong because the *base* is not the same for the increase as for the decrease. Suppose the population of a certain city at one specified date was 25,000 and during a ten-year period the population increased 10 per cent. We write

$$\frac{\text{Per cent increase}}{25{,}000} = \frac{10}{100} \quad \text{or} \quad \text{Per cent increase} = 25{,}000 \times \frac{10}{100} = 2500$$

The new population is then $25{,}000 + 2500$ or 27,500. Now assume during the next ten-year period the population *decreased* 10 per cent. To find the population at the end of this ten-year period,

$$\frac{\text{Per cent decrease}}{27{,}500} = \frac{10}{100} \quad \text{or} \quad \text{Per cent decrease} = 27{,}500 \times \frac{10}{100} = 2750$$

The new population is then $27{,}500 - 2750 = 24{,}750$. This, of course, is not the same as the original population of 25,000. Here again if one keeps in mind the fact that *ratios* are involved and that the *base* may not remain the same, no difficulty need arise.

Another common error in the use of per cent occurs in speaking of something as "150 per cent larger." This is too often intended to mean that the increase is *by* 50 per cent *to* 150 per cent of the original value; clearly unfortunate usage.

One last example of misuse of per cent will be cited. A number A is given as, say, 90 per cent of a number B. What is the number B? The value of B is *not* obtained by adding 10 per cent to A as is rather commonly done.

If we write the ratio

$$\frac{A}{B} = \frac{90}{100}, \text{ we obtain } B = \frac{A}{90} \times 100$$

as the value of B. We recognize here that A is a percentage of a number B which is the *base*, and the per cent that A is of B is 90. Thus we can write the above ratios.

Care is required to distinguish the base for several common uses of the term per cent, as follows. The percentage *accuracy* of a meter is the ratio, in per cent, of its registration to the true value. The percentage *error* of a meter is its percentage accuracy minus 100. Each of these obviously has *true* value as base. The percentage *correction*, however, is the ratio, in per cent, of error to *registration*. Evidently the per cent of error and the per cent of correction will have opposite signs and, due to the different base, will not have the same value although the discrepancy will be small for a small per cent of error or correction. It should be observed further that the per cent of error discussed earlier for an inexact number is based not on the true value, as is the per cent of error for a meter, but on the most probable value of the quantity.

15. ***Graphs.*** Graphs are usually plotted on paper ruled according to one of four systems, viz.,

1. Plain cross section.
2. Log-log.
3. Semi-log.
4. Polar.

These are available in several sizes and weights of sheets, spacing of lines, color of ink, etc. The most useful for engineering is easily item 1 of the above, mostly in an 8½ in. by 11 in. size ruled either 20 lines per inch or in millimeters. Every tenth line is usually distinguished by heavier weight, and every fifth line may be of intermediate weight. It is rare in engineering that rulings in other than these five and ten groupings are used.

Item 2, the log-log, has each coordinate ruled in the same manner as the *ABCD* scales of a slide rule and also without regard for decimal points. These are designated two by three, three by four, etc., indicating the number of log cycles (1 to 10) for each coordinate. This paper is used mostly for plotting experimentally determined data which, for these scales, may give straight lines. The function $ay = x^n$, for example, in log-log form ($\log ay = n \log x$) reduces to a straight line of slope n. The alignment of the plotted data as a straight line is far more readily checked than is the conformity of the data to the curved plot obtained on plain graph paper. Furthermore, the log-log plot facilitates determining the value of the exponent n; it is the *actual* slope of the plotted line as measured by *linear* (not the log) scales. A quite different application of log-log paper is found on page 175, Fig. 11·2 where, owing to the nature of μ, a third axis at 45° (unity slope) becomes possible. This example also illustrates how the characteristic compression of the higher scale values into much less space than required for plain linear scales can be advantageous.

The semi-log (item 3) is a cross between the two just considered, having one coordinate linear and the other logarithmic. It is used either to compress the higher values and expand the lower values of one coordinate, as suggested for H in Chapter XI, Art 10, or to obtain a linear plot of an exponential function such as $y = A\epsilon^{ax}$ which gives $\log (y/A) = ax$. These occur in Chapters XV, XVII, and XVIII although no semi-log plots are utilized there.

The polar plot (item 4) is used to show the angular distribution of candle-power from a source of light, and for some types of recording instruments which rotate the circular chart in proportion to time. It does not find as extensive application as the rectangular types.

16. Scales for Graphs. The choice and layout of scale are governed by three considerations:

1. Range of data.
2. Accuracy of data.
3. Readability of interpolations.

Clearly the length of scale should be sufficient to cover the range of data to be plotted but it should not greatly exceed this length. When the data do not include zero or near-zero values it should not too hastily be presumed that zero need not appear on the graph. Absence of the origin of coordinates is but *occasionally* desirable; it is usually essential either for operations (possibly graphical constructions) to be performed with the graph or simply for conveying sense of proportion in the variations of quantities, possibly as ratios rather than as mere differences or increments. For the log scales this question of course is removed by the non-existence of a zero value.

It is important (item 2) that the accuracy with which the scale can be read be compatible with the accuracy of the data. To plot a value 15.3, where 3 is doubtful, to a scale which permits 15.37 to be read with only the 7 doubtful, exaggerates the apparent accuracy of the data. To use too small a scale is no less undesirable because it fails to utilize the available accuracy of the data.

Readability of interpolations (item 3) requires special attention both because it is important and because it is so often ignored. If a scale is placed on a graph only in accord with the criteria of items (1) and (2), it is more than possible that the sub-intervals between marked integer values will be 3, 6, 7, etc., which decimally represent awkward values and interfere seriously with both the plotting and reading of the graph. It is always preferable to compromise with the requirements of items (1) and (2), when necessary, to insure that only the sub-intervals 2, 4, 5, 10, 20, etc., are permitted, so that their decimal significance will be respectively 0.5, 0.25, 0.2, 0.1, 0.05, etc. Of these, the 4 interval, of course, is least desirable although it is usually considered acceptable.

For economy of space or other reasons, more than one scale may be placed on either or both axes. In this event consideration should be given to the possibility of confusion because one curve may come so close as to tangle with another. Avoidance of this influenced the choice of scales in Fig. 5·4, page 76. Identification of each curve with its proper scales is imperative under these conditions. In the instance just cited this is done with letters. Where only *one* axis carries

more than one scale, like Fig. 11·1, page 174, the more satisfactory scheme of using different kinds of line (solid, dotted, dashed, etc.) is feasible. For Fig. 12·3, pages 186–187, however, which is used extensively in the solution of problems, it was considered worth while to avoid entirely the inevitable hazard of error which dual scales introduce. When the limitation of single color reproduction is not imposed, as it is for blueprinting, mimeographing, and for this text, the use of color for distinguishing curves (and possibly scales) is advantageous.

Not only must the scale be identified with the curve concerned, but the significance of each scale of values must be clearly indicated, including *units*, and if in per cent or per unit values, the *base* is important. When too bulky for placing at the scales, these data, in part at least, may be included under the title of the graph. For engineering work the title is always a plain unadorned legend devoid of the flourishes sometimes found on old maps and manuscripts. It should be placed to avoid interfering with any of the curves and should read in the same direction as the abscissa. The abscissa, of course, is the horizontal and the ordinate is the vertical coordinate; these lines should be clearly re-ruled. It is usual to scale the plus values of coordinates to increase from the origin to the right and upward, but occasionally there is reason to violate this rule. This occurs, for example, when two scales refer to the same curve, but one (possibly a reciprocal) must decrease in the direction for which the other increases. The four quadrants are sometimes used so that each quadrant involves a different pair of scale units; the four scales are then likely to increase outwardly from a common origin or zero value and have no minus values. Although it is usual to scale values of the independent variable along the abscissa and values of the dependent variable along the ordinate, this too might be violated in the four quadrant exception just cited.

PROBLEMS

1. The current I in a circuit is measured from the voltage E across a known resistance R in the circuit; $I = \dfrac{E}{R}\cdot$ $R = 1.0061 \pm 0.0040$ ohm. If it were desired to determine I within 0.5 per cent, how precisely would E have to be measured?

2. The area A of a triangle whose base b is about 10 cm and height h about 32 cm is to be determined within 0.30 per cent. Assuming equal effects, what are the allowable percentage deviations in b and h?

3. The volume of a sphere whose diameter is about 6 in. is desired within 0.10 per cent. How closely must the diameter be measured?

4. Find the sum of the following lengths: 1.253, 24.71, 1.394, 0.6159, 0.01286. All lengths are given in feet.

5. A section of land is 19 ft long and 51.7 ft wide. What is the area?

6. A rope was cut into 19 pieces each piece being 51.7 ft long. How long was the rope?

7. Given to find $37.6 \times 0.582 \times 2.36 \times 242$, describe the steps required:

(*a*) Using only the C and D scales.

(*b*) Using also the CI-scale.

(*c*) Show how to determine the decimal point.

8. Given $C = \sqrt{A^2 + B^2}$:

(a) For $A = 3B$, compute the per cent error in C when C is computed by the approximate method.

(b) What minimum ratio A/B will permit use of the approximate method when C is required to have at least four significant figures?

9. Given a right triangle with hypotenuse 58.2 and one side 2.53:

(a) Compute the other side by the approximate method.

(b) Describe the slide rule steps required to find the smallest angle and give its value.

10. Measurements are taken which show

$$R = 1356$$
$$I = 78.0$$

Compute the value of $E = RI$ and express it in such manner that the number of significant figures for E will not be ambiguous.

11. The resistance of a wire is expressed by

$$R = \rho \frac{l}{A}$$

where R = resistance of wire in ohms.
ρ = resistivity in ohm-centimeters.
l = length of wire in centimeters.
A = cross section of wire in square centimeters.

If the resistance of a wire is to be reduced 23 per cent and its length increased 18 per cent, how must its cross section be changed? ρ is constant.

12. (a) What number decreased by 15 per cent of itself equals 255?

(b) 1200 is 25 per cent less than what number?

13. A student sold a car at a gain of $48.62, which was 17 per cent of the cost. What was the cost and the selling price?

14. A watthour meter that ran 2.5 per cent too fast registered the passage of 65,600 kilowatthours of electrical energy in a year. What was the *actual amount* of electrical energy that passed through the meter?

15. The correction factor to be applied to a voltmeter which reads too high between 15 and 20 volts is found to be 3.0 per cent. The meter is used to measure 18 volts. What per cent of true value will the meter read?

16. Three electrical machines are connected together so that the output of one is the input to the next, etc. The losses of the first machine are 10 per cent, of the second, 15 per cent, and of the third, 20 per cent. What are the total losses (in per cent) of the set as a whole?

17. It is desired to compute the period T of a pendulum from its length and the value of g.

$$T = 2\pi \sqrt{\frac{l}{g}}$$

If $l = 107.31$ cm known within 0.2 per cent, and
$g = 980.63$ cm/sec^2 known within $\frac{1}{10}$ per cent
what will be the uncertainty in T in per cent? How many significant figures should be used in computing T?

18. The resistance R_0 of a certain coil at 0° C is 30.010 ohms, known within 0.015 ohm. Its resistance at any other temperature t is given by the relation

$$R_t = R_0(1 + \alpha t)$$

where α, the temperature coefficient, is 0.00491, known within 1.0 per cent. If a thermometer is used which can be depended upon within 0.3° C, how precisely, expressed in percentage, can the resistance at 45° C be computed?

19. The deflection of the electron stream in a magnetically controlled cathode-ray oscilloscope is expressed by

$$d = KBV^{-\frac{1}{2}}l_1(\tfrac{1}{2}l_1 + l_2)$$

(*a*) What per cent change in d will a 5 per cent increase in voltage V produce?

(*b*) If $l_2 = 32l_1$, what error in the value of d will an error of 3 per cent in the measurement of l_1 produce?

20. Given: $C = 53.7$, $B = 7.62$. Determine $A = \sqrt{C^2 - B^2}$ by using the decitrig or vector rule. Describe the steps used in obtaining your result.

21. The equation for the torque developed by an electric motor is

$$T = K\phi I_a$$

where T is the torque in pounds-feet, I_a the armature current of the machine in amperes, ϕ the magnetic flux in the machine in lines, and K, a constant depending on the construction of the machine. What curve will be obtained on plotting T versus I_a? Sketch the curve and justify its intercepts.

22. The following equation applies to a d-c motor:

$$S = \frac{E_t - I_aR_a}{K\phi}$$

where S is the speed in rpm, E_t the voltage applied to the machine, I_a the armature current in amperes, R_a the armature resistance in ohms, ϕ the magnetic flux in the machine in lines, and K a machine constant. A test is run in the laboratory in which I_a, R_a, ϕ, and K are all constant in value. From the data a curve of S versus E_t is plotted. Sketch the curve, and explain how you would obtain the value of I_aR_a graphically.

23. The equation for the losses in the iron of an electrical machine takes the form

$$P = K'f + K''f^2$$

where P is the power converted into heat in the iron in watts, f the frequency in cycles per second, and K' and K'' constants. Rewrite the equation so that it may be plotted as a straight line with frequency as abscissa. Show how the values of K' and K'' may be obtained from the curve.

24. The equation for the emission current from a heated filament is

$$i = AT^2\epsilon^{-\frac{b_0}{T}}$$

where i is the current in amperes per square centimeter of heated surface, T the temperature in degrees Kelvin, and A and b_0 constants. Show that this equation may be rearranged to give a straight line plot, giving quantities to be plotted as ordinates and as abscissas, and the kind of cross-section paper to be used. Indicate also how the value of b_0 may be obtained from such a plot.

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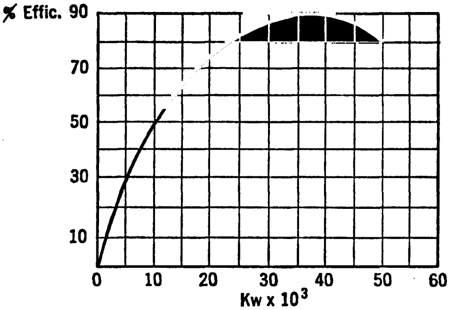
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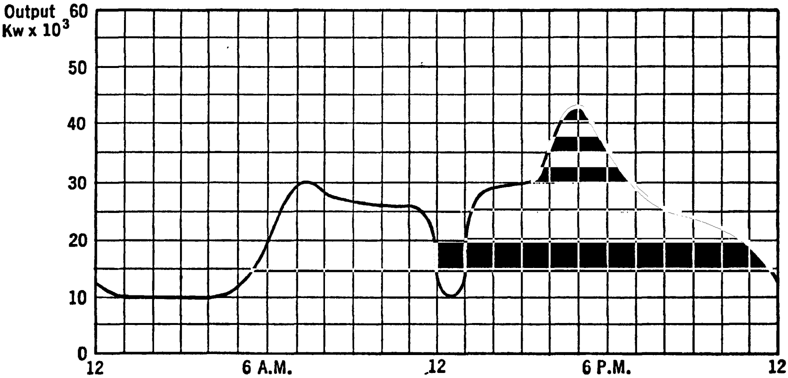
Kw x 10^3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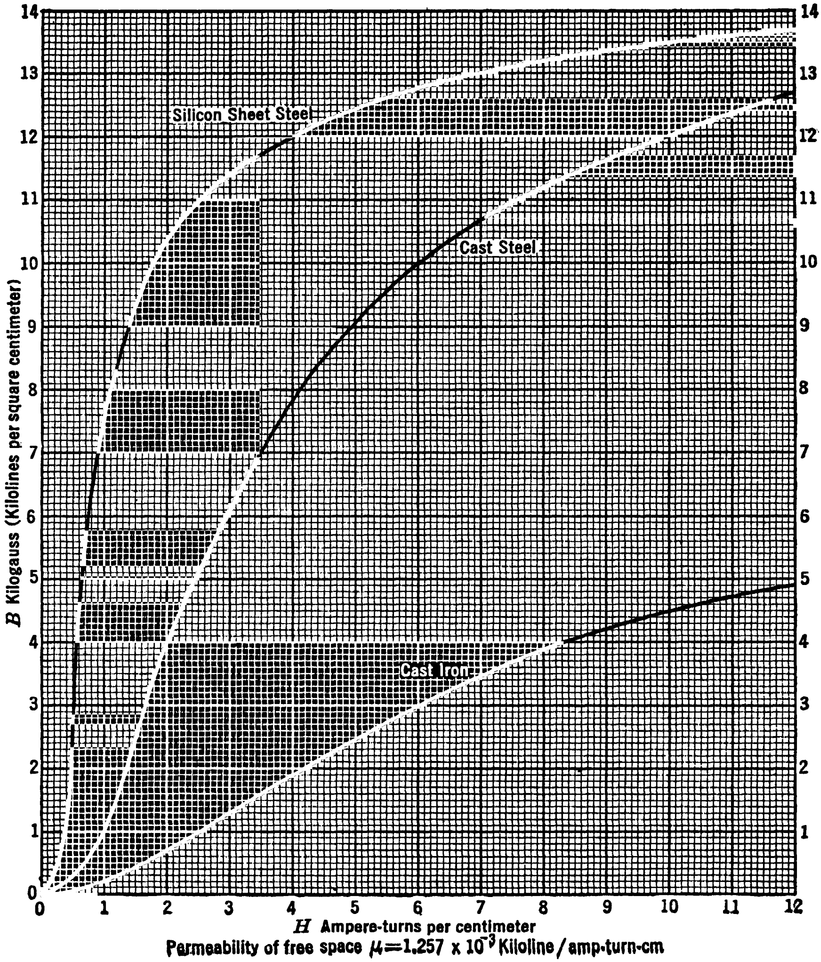
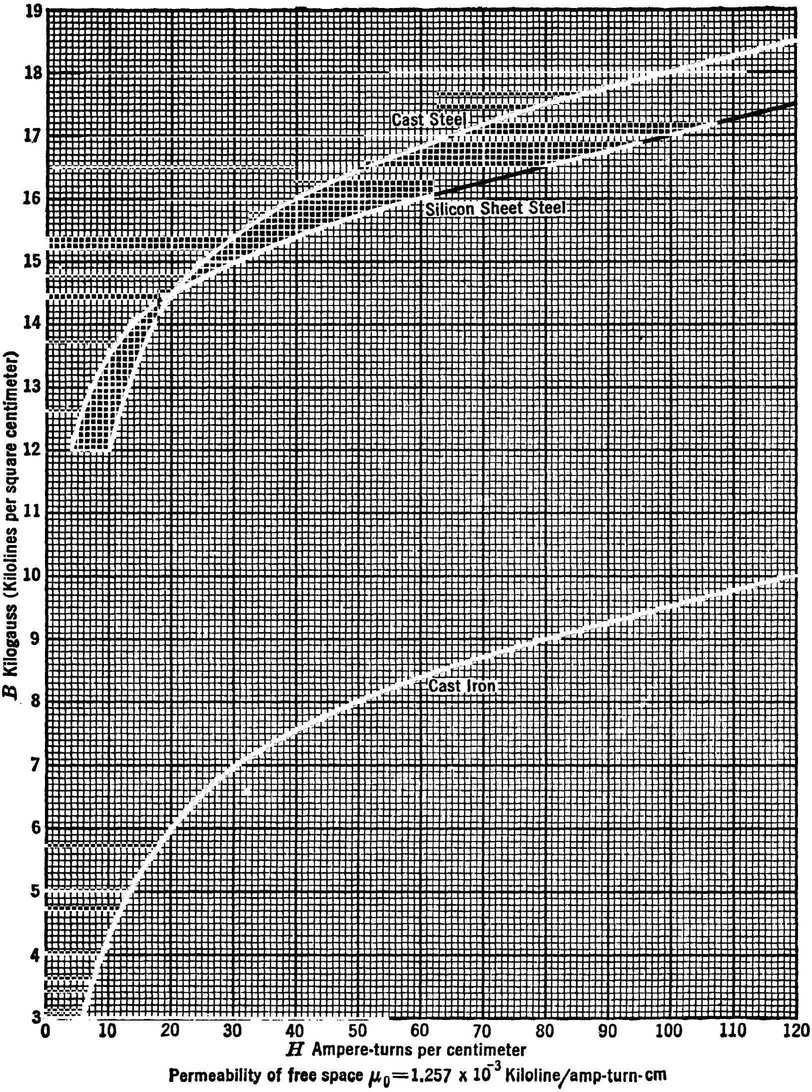
6 P.M.
12



Silicon Sheet Steel
Cast Steel
Cast Iron
B Kilogauss (Kilolines per square centimeter)
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
0
1
2
3
4
5
6
7
8
9
10
11
12
H Ampere-turns per centimeter
Permeability of free space $\mu = 1.257 \times 10^{-3}$ Kiloline / amp-turn-cm



Permeability of free space $\mu_0 = 1.257 \times 10^{-3}$ Kiloline/amp-turn-cm

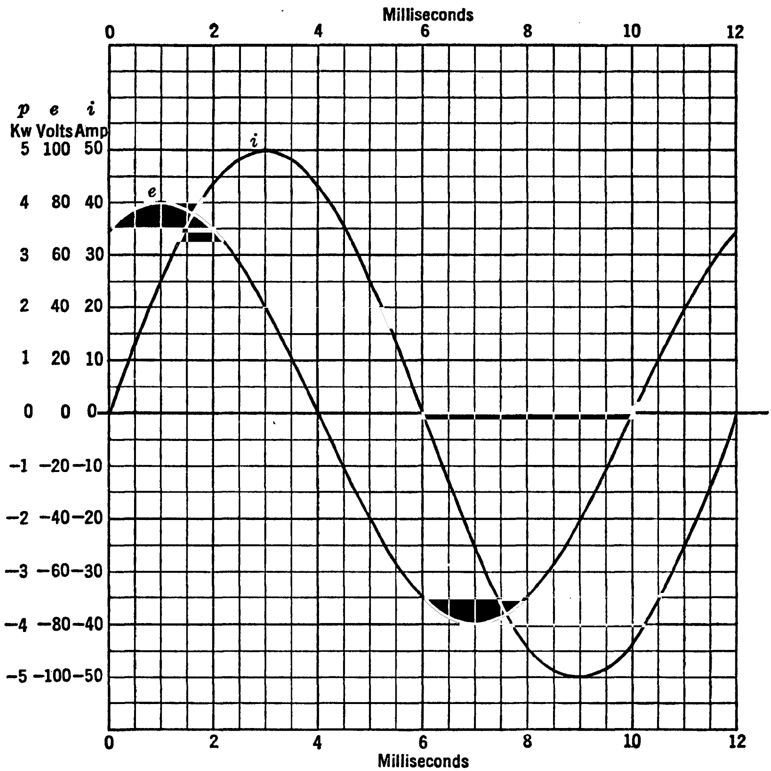
Design of Lifting Magnet

For air gap, B = ______________ kl per sq in. $\frac{w}{d}$ = ______________

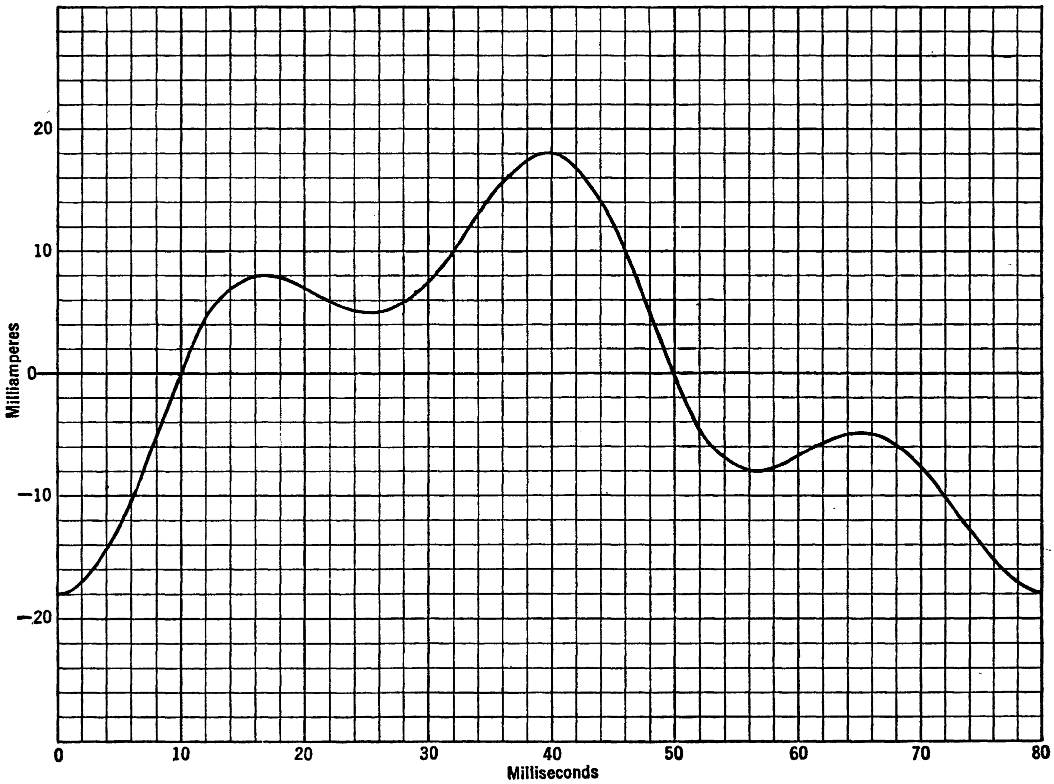
	Trial	Final
$F_1 = F_2$ (lb)		
$A_1 = A_2$ (sq in.)		
d (in.)		
w (in.)		
$D = d + 2w$ (in.)		
$D_m = d + w$ (in.)		
t (in.)		
h (in.)		
$h' = 0.8h$ (in.)		
a (in.)		
B_{cs} (yoke) (kl per sq in.)		
ϕ (kl)		
Armature densities (plot)		
B_a (kl per sq in.)		
B_b (kl per sq in.)		
B_c (kl per sq in.)		
B_d (kl per sq in.)		
B_e (kl per sq in.)		
B_f (kl per sq in.)		
B_g (kl per sq in.)		
Armature gradients (plot)		
H_a (amp-turns per in.)		
H_b (amp-turns per in.)		
H_c (amp-turns per in.)		
H_d (amp-turns per in.)		
H_e (amp-turns per in.)		
H_f (amp-turns per in.)		
H_g (amp-turns per in.)		
Average gradients		
Yoke (amp-turns per in.)		
Air gaps (amp-turns per in.)		
Mean Flux Path Lengths		
Yoke (in.)		
Air gaps (total) (in.)		
Armature (in.)		
Mmf (Hl's)		
Yoke (amp-turns)		
Air gaps (total both) (amp-turns)		

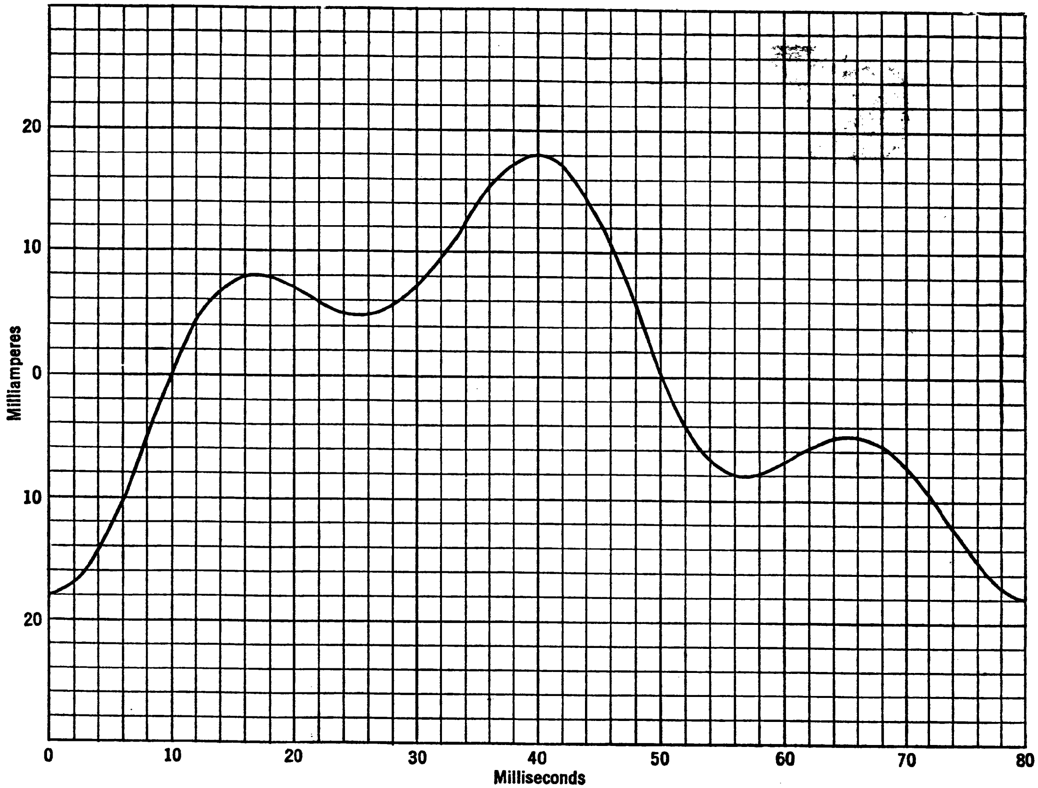
	Trial	Final
Mmf (Hl's) (continued)		
Armature (from graph) (amp-turns)		
Total (amp-turns)		
l_m (length of mean turn, in.)		
ρ (copper at 65° C) (cir mil ohms per ft)		
E (volts)		
A (computed), (cir mils)		
A.W.G. wire		
A (actual), (cir mils)		
Coil section ($w \times h'$), (in^2)		
Space factor		
Copper section, (cir mils)		
Coil turns		
R for coil (actual) (ohms)		
I (amp)		
P (watts)		
Cooling surface		
Cylindrical coil surface (sq in.)		
1 Coil end (sq in.)		
Total (sq in.)		
Watts per sq in. (to be dissipated)		
Cooling coefficient watts per in^2-°C		
Temp. rise of coil surface °C		
Volume of CS, cu in.	Compute only for final design	
Weight of CS, lb		
Cost per lb of CS		
Cost of CS		
Length of wire (ft)		
Weight of copper		
Cost per lb of copper		
Cost of copper		
Total cost of metal		

Computed by .



Milliseconds
0
2
4
6
8
10
12
p e i
Kw Volts Amp
5 100 50
4 80 40
3 60 30
2 40 20
1 20 10
0 0 0
−1 −20 −10
−2 −40 −20
−3 −60 −30
−4 −80 −40
−5 −100 −50
i
e
Milliseconds


Milliamperes
20
10
0
−10
−20
0
10
20
30
40
50
60
70
80
Milliseconds


Milliamperes
20
10
0
10
20
0
10
20
30
40
50
60
70
80
Milliseconds